Anti-Specker Properties in Constructive Reverse Mathematics

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Abstract

Constructive reverse mathematics is a programme in which non- and semi-constructive principles are classified in accordance with which other principles they imply or are implied by, relative to the framework of Bishop-style constructive mathematics. One such principle that has come under focus in recent years is an antithesis of Specker's theorem (that theorem being a characteristic result of Russian recursive mathematics): this so-called *anti-Specker property* is intuitionistically valid, and of considerable utility in proving results of real and complex analysis.

We introduce several new weakenings of the anti-Specker property and explore their role in constructive reverse mathematics, identifying implication relationships that they stand in to other notable principles. These include, but are not limited to: variations upon Brouwer's fan theorem, certain compactness properties, and so-called *zero-stability* properties. We also give similar classification results for principles arising directly from Specker's theorem itself, and present new, direct proofs of related fan-theoretic results.

We investigate how anti-Specker properties, alongside power-series-based arguments, enable us to recover information about the structure of holomorphic functions: in particular, they allow us to streamline a sequence of maximum-modulus theorems.

Keywords: anti-Specker property; constructive mathematics; constructive reverse mathematics; fan theorem; holomorphy; maximum-modulus theorem; power series; zero-stability.

Preface

The aim of this thesis is to explore the role of several antitheses of Specker's theorem in constructive reverse mathematics (CRM). In particular, we will establish — in as direct a manner as possible — results that show how these principles can be used in real and complex analysis.

The first two chapters lay the groundwork for our subsequent investigation. In particular, Chapter 1 sets the philosophical stage of constructive mathematics, and in Chapter 2 we cover some of the preliminary technical details and themes that will occur repeatedly throughout the following work. In doing so, we will also take a brief look at some of the most significant principles of the CRM programme, and summarise the relationships between these.

The novel content of this thesis may be found in chapters 3 through 6. We begin by introducing several so-called *anti-Specker properties* and seeing how they relate to other principles of real analysis (most notably including two Heine-Borel properties). Alongside these anti-Specker properties, we similarly consider the related *limit-stability property*.

Chapter 4 continues this investigation with a focus upon principles provable within the recursive framework of **RUSS**. In particular, our analysis here revolves around equivalence classes for the *Specker property* and its increasing counterpart. Contraposition of these results yields (negative) implications that tie in with those of Chapter 3.

In Chapter 5, we make a brief departure from our study of anti-Specker properties to examine relationships between *omniscience principles* and variations upon *Brouwer's fan theorem*. While the results of this chapter are known, the proofs we employ are new and direct, and thus interesting in their own light.

Chapter 6 carries our investigation over into the arena of complex analysis. We observe how power series expansions may be used to produce relatively simple proofs about the location of zeroes of holomorphic functions, and then use anti-Specker properties to extend a similarly-proved maximum-modulus result to a much more powerful formulation. While this resulting maximum-modulus theorem is known, these anti-Specker properties allow us to produce a new, more streamlined proof: in particular, we do not require *Cauchy's integral formula*. Finally, we move to a more general setting and study the property of *zero-stability* as it relates to our anti-Specker properties.

1 | Constructive Mathematics

1.1 Algorithmic Proof

It is a truism of mathematics that not all proofs are created equal. While the payload of a proof is the theorem that it establishes, a proof that is particularly interesting will often have further value in that its body sheds light upon the relationships between the objects it incorporates. We speak of such proofs as having *algorithmic content* if they embody deterministic procedures by which one can (in principle) construct the objects of the proof, in a finite amount of time. The most immediate illustration of algorithmic content is the existence proof: to show that an object x exists with the property $P(x)$, one could assume that no such x exists and derive a contradiction. However, this approach typically yields very little information about x: a better approach (if possible) would be to give an argument showing how to *construct* an object x satisfying $P(x)$. We would then be able to actually *use* this object by (say) reconstructing it within some computer program that depends upon the property P.

Constructive mathematics is an approach to mathematical enquiry that allows only those proof techniques that yield algorithmic content. Accordingly, constructive proofs are often more contentful than their counterparts in standard nonconstructive (*classical*) mathematics. In shifting the focus from the outcome of a proof to its contents, we change what is meant by the assertion of a sentence S. Classically, in the assertion of S, one claims that S is *true*; constructively, however, the assertion of S instead corresponds to the claim that *there is a proof of* S.

Under this interpretation, one must adopt a different understanding of the various logical components that structure the sentences of mathematics. The *BHK (Brouwer-Heyting-Kolmogorov)* *interpretation* of logical connectives and quantifiers specifies (inductively) what it means to be a proof of any given proposition, by interpreting its constituent parts as follows [BV06, p. 8].

- A proof of $P \wedge Q$ consists of a proof of P and a proof of Q.
- A proof of $P \vee Q$ consists of either a proof of P or a proof of Q (and we know which).
- A proof of $P \implies Q$ is an algorithm that can convert any proof of P into a proof of Q.
- The absurdity $\perp$ has no proof.
- A proof of $\neg P$ (defined as $P \implies \perp$) is an algorithm that can convert any proof of P into a proof of $\perp$ (that is, an absurd statement) [Tro91, p. 12].
- A proof of $(\exists x \in X)P(x)$ is an algorithm that computes an object x and demonstrates both that $x \in X$, and that $P(x)$ holds.
- A proof of $(\forall x \in X)P(x)$ is an algorithm that, given any object x and a proof that $x \in X$, demonstrates that $P(x)$ holds.

Because of this, there are logical principles of classical mathematics that we must reject constructively. Consider *double negation elimination* (**DNE**): the principle that, for any proposition P, one can deduce P from $\neg\neg P$. Under the BHK interpretation, there is no general way to produce a proof of P from a proof that it is impossible for P to be impossible; hence this principle does not hold constructively. **DNE** is therefore not a valid principle of the *intuitionistic logic* that underlies constructive mathematics (characterised by the BHK interpretation); indeed, this logic is equivalent to classical logic without **DNE**.

Consider also the *law of excluded middle* (**LEM**), which states that for every proposition P, either P or $\neg P$ holds. One can easily show that this principle implies **DNE** over intuitionistic logic; hence we reject it constructively. (This fits well with our shift of focus from truth to provability: given a proposition P, we are not necessarily able to find either a proof of P or a proof that P is impossible.) It is by this rejection of **LEM** that constructive mathematics is frequently (mistakenly) characterised.

Another benefit of working constructively is that the language of constructive mathematics is *richer* than that of classical mathematics, in the sense that it allows us to make distinctions not possible classically. For example, we can distinguish *positive assertions* such as $A \vee B$ from classically equivalent *negative assertions* such as $\neg(\neg A \wedge \neg B)$. A deeper example concerns the notion of compactness, which we will return to in Section 2.2: the three major characterisations of compact metric spaces (completeness and total boundedness; sequential compactness; the Heine-Borel property[1]), while classically equivalent, are constructively distinct.

1.2 A Brief History

Mathematics before the late 19$^{\text{th}}$ century was for the most part constructive, in the sense that proofs of existence were required to demonstrate how to find the objects to which they pertained [Tro91, p. 1]. The genesis of widespread nonconstructivity in mathematics was marked by David Hilbert's 1888 proof of his *finite basis theorem*, in which he demonstrated the existence of a finite basis for each ideal in a particular ring by deriving a contradiction from the assumption of its nonexistence [Ros12, p. 99]. His contemporary Paul Gordan famously said of this proof: "This is not mathematics. This is theology.[2]"

Hilbert's proof arose against a background of movement towards increased abstraction in mathematics, led by giants such as Georg Cantor and Richard Dedekind. Cantor, in particular, granted unprecedented legitimacy to infinitary concepts with the introduction of his set theory and subsequent transfinite arithmetic [Pos98, pp. 293–294]. The themes of constructivism first emerged as a response to this increasing level of abstraction with the views of Leopold Kronecker. Kronecker was a *finitist*: he believed that only the natural numbers and those structures that could be finitely represented in their terms were valid objects of mathematics, and in doing so opposed Cantor's set theory [TvD88, p. 2]. He also required that existence proofs embody constructions, in the spirit of today's constructive mathematics.

This greater degree of abstractness — along with later concern about the well-foundedness of set theory, raised by considerations such as Russell's paradox — prompted a foundational crisis around the turn of the 20$^{\text{th}}$ century that would last for some decades. Gottlob Frege in particular identified the lack of a satisfactory theory of the nature of the objects of mathematics, or indeed mathematical

[1]See Section 3.2.
[2]"Das ist nicht Mathematik. Das ist Theologie." [Rei96, p. 34]

enquiry itself [Dum77, p. 1]. Mathematics seemed to stand in need of justification, and it was to this need that Hilbert and his contemporary L. E. J. Brouwer responded, igniting a controversy.

Hilbert attempted to secure the entirety of mathematics by dividing it into two parts [Pos98, pp. 294–295]. On the one hand, there was the unproblematic, finitary part of mathematics, consisting of finite arithmetic and combinatorics, which Hilbert believed to be grounded in intuition. On the other were the more abstract fields such as set theory, which he referred to as the "ideal" part of mathematics. Hilbert sought to show that the ideal branches of mathematics were reliable by reducing them to a formal axiomatisation, and then using finitary mathematics to prove that the resulting formal system was consistent. (This programme would later be proved impossible by Gödel's second incompleteness theorem.)

Brouwer took a different approach: rather than seeking to justify the entirety of classical mathematics, he interpreted the foundational crisis as an indication that something was amiss with the way mathematics was done [Dum77, p. 2]. He believed that mathematics should stand as its own justification; accordingly, he maintained that logic was a part of mathematics, rather than a foundation for it. Brouwer's student Arend Heyting wrote [Hey71, p. 6]: "A mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever."

The goals of Brouwer's resulting programme, *intuitionism*, were twofold: he sought both to build up mathematics according to his own principles, and to present a critique of the nonconstructivities of classical mathematics [Bee85, p. 430]. He conceived of mathematics as being a languageless activity of the mind, arising ultimately from the intuition of time [Bro81, p. 4]. Thus, in his view, the objects of mathematics were mental constructions, and to prove the existence of such an object was to give its construction.

1.3 Brouwerian Counterexamples

A cornerstone of Brouwer's critique of classical mathematics was a new type of counterexample, now called a *Brouwerian counterexample*. Rather than demonstrating the falsity of a principle, a Brouwerian counterexample shows that, when interpreted constructively, the relevant principle would allow one to find solutions to a variety of unsolved mathematical problems and hence must be rejected. A widely-used example of such a problem is the Goldbach conjecture, which states that every even number greater than 2 can be written as the sum of two primes: it is not currently known whether this claim is true or false.

The following example is a variation upon an argument of Brouwer himself [Bro23, p. 337].

Proposition 1.1: The following *trichotomy law* cannot hold constructively:

($*$) For every real number x, either $x = 0$, $x > 0$ or $x < 0$.

Proof. Suppose that ($*$) holds. Define a sequence $(x_n)_{n \geqslant 1}$ by

(i) $x_n = \left(-\frac{1}{2}\right)^k$ if, for some minimal k with $2 \leqslant k \leqslant n$, the number $2k$ cannot be written as the sum of two primes; and

(ii) $x_n = \left(-\frac{1}{2}\right)^n$ otherwise.

The sequence (x_n) converges to some real number x. Then by ($*$), we can decide which of the following cases holds:

- $x = 0$, whence case (ii) holds for all n and the Goldbach conjecture is true; or
- $x > 0$ or $x < 0$, whence case (i) holds eventually and the Goldbach conjecture is false.

But this allows us to decide the truth or falsity of the Goldbach conjecture; hence ($*$) cannot be constructively valid. □

Note that the use of the Goldbach conjecture here is *ad hoc*: this counterexample can be re-cast in terms of any of a multitude of open problems. Brouwer himself used the location of the string of digits 0123456789 in the decimal expansion of π to fill this role.[3]

In later years, Errett Bishop [Bis67, p. 9] and others identified and abstracted (from examples given by Brouwer) several incontrovertibly nonconstructive principles. This gave rise to a more systematic

[3]It is now known that such a string occurs for the first time from the 17 387 594 880$^{\text{th}}$ digit after the decimal point [Bor98, p. 14].

form of Brouwerian counterexample: rather than showing that a principle implies a solution to an arbitrarily chosen unsolved problem, the aim became to prove that it entails one of these so-called *omniscience principles*.

1.4 Omniscience Principles

These embody *potentially infinite searches*; hence the use of one in a proof introduces an essentially non-algorithmic step into the argument. The strongest and most fundamental of these principles is the (full) *principle of omniscience* [Bis67, p. 9]:

PO For each set A and property P, either all elements of A have the property P or there exists an element of A with the property $\neg P$.

Bishop believed that the use of this principle was responsible for most of the nonconstructive aspects of classical mathematics. However, it rarely appears in the modern constructive mathematics literature, as several of its weakenings serve to more precisely capture the nature of most of the nonconstructivities the mathematician encounters. The most immediate such weakening is the *limited principle of omniscience*, identified by Bishop himself [Bis67, p. 9]:

LPO For each binary sequence $(\lambda_n)_{n\geqslant 1}$, either all the terms are equal to 0, or else there exists a term equal to 1.

$$(\forall n)\left[\lambda_n = 0\right] \vee (\exists n)\left[\lambda_n = 1\right]$$

The other omniscience principle presented by Bishop is the weaker *lesser limited principle of omniscience*:

LLPO For each binary sequence $(\lambda_n)_{n\geqslant 1}$ with at most one term equal to 1, either all the even terms of the sequence are equal to 0, or else all the odd terms of the sequence are equal to 0.

$$\begin{aligned}(\forall m)(\forall n)&\left[m \neq n \implies \lambda_m\lambda_n = 0\right]\\ &\implies \Big((\forall n)\left[\lambda_{2n} = 0\right] \vee (\forall n)\left[\lambda_{2n-1} = 0\right]\Big)\end{aligned}$$

Also of interest is the *weak limited principle of omniscience*:

WLPO For each binary sequence $(\lambda_n)_{n\geqslant 1}$, either all the terms are equal to 0, or it is impossible for all the terms to be equal to 0.

$$(\forall n)\left[\lambda_n = 0\right] \vee \neg(\forall n)\left[\lambda_n = 0\right]$$

Despite being in part a negative result — rather than an entirely positive one like **LLPO** — the principle **WLPO** is in fact stronger than **LLPO**. Indeed, we have (constructively[4]) the following implications [BR87, p. 5]:

$$\textbf{PO} \implies \textbf{LPO} \implies \textbf{WLPO} \implies \textbf{LLPO}.$$

These four omniscience principles are all provably false in two of the schools of constructive mathematics we will encounter in this thesis;[5] hence their constructive inviability is clear [BR87, p. 4]. However, there is a fifth principle whose status is less clear-cut: *Markov's principle* (of unbounded search):

MP For each binary sequence $(\lambda_n)_{n\geqslant 1}$, if it is impossible for all the terms to be equal to 0, then there exists a term equal to 1.

$$\neg(\forall n)\left[\lambda_n = 0\right] \implies (\exists n)\left[\lambda_n = 1\right]$$

Unlike the other omniscience principles, **MP** does not represent a possibly infinite search. Its premiss states that the sequence we apply it to cannot contain only zeroes; we therefore know that, should we examine successive terms of (λ_n), we are guaranteed to eventually encounter a term equal to 1 and can thus determine its index. Hence there is a sense in which **MP** is algorithmic. The problem, however, is that it does not specify a *bound* on how long one would have to examine the sequence for in order to find the term equal to 1: this process could take longer than the lifetime of the universe.

[4]That is, relative to **BISH**; see Section 1.6.
[5]That is, **INT** and **RUSS**; again see Section 1.6.

1.5 Axioms of Choice

We now come to another family of principles whose acceptance or rejection is an issue close to the heart of constructive mathematics. The full *axiom of choice* states:

AC Let A and B be sets, and $S \subseteq A \times B$. Suppose that, for each $x \in A$, there exists $y \in B$ such that $(x, y) \in S$. Then there exists a function $f\colon A \to B$ such that $(x, f(x)) \in S$ for each $x \in A$.

It has been shown [GM78, p. 461] that this axiom implies **LEM** within intuitionistic set theory; hence it cannot be a part of any system of constructive mathematics. There are, however, three weakenings of **AC** that are often accepted constructively. The axiom of *dependent choice* states:

DC Let A be a set, and take $a \in A$ and $S \subseteq A \times A$. Suppose that for each $x \in A$, there exists $y \in A$ for which $(x, y) \in S$. Then there exists a sequence $(a_n)_{n \geqslant 1}$ of elements of A such that $a_1 = a$ and $(a_n, a_{n+1}) \in S$ for each $n \in \mathbf{N}^+$.

(Here, $\mathbf{N}^+ \equiv \{1, 2, 3, \dots\}$ stands for the set of positive integers.) This implies the weaker axiom of *countable choice*, which is just the full axiom of choice with A replaced by the set $\mathbf{N} \equiv \{0, 1, 2, \dots\}$ of natural numbers:

AC$_\omega$ Let B be a set, and $S \subseteq \mathbf{N} \times B$. Suppose that for each $n \in \mathbf{N}$, there exists $x \in B$ such that $(n, x) \in S$. Then there exists a function $f\colon \mathbf{N} \to B$ such that $(n, f(n)) \in S$ for each $n \in \mathbf{N}$.

Finally, we have the axiom of *unique choice*, also known as the axiom of *nonchoice*. This identifies the function f with its graph, S:

AC! Let A and B be sets, and $S \subseteq A \times B$. Suppose that, for each $x \in A$, there exists a *unique* $y \in B$ such that $(x, y) \in S$. Then there exists a function $f\colon A \to B$ such that $(x, f(x)) \in S$ for each $x \in A$.

In these three weaker cases, the existence of the choice function f follows directly from the constructive interpretation of the quantifiers. Why is this not also true of the full axiom of choice?

Consider the premiss of **AC**,

$$(\forall x \in A)(\exists y \in B)\big[(x, y) \in S\big].$$

Under the BHK interpretation, a proof of this fact is an algorithm that, given any object x and a proof that $x \in A$, constructs an object y and demonstrates that $y \in B$ and $(x, y) \in S$ hold. We seek to extract a choice function from this algorithm; however, the algorithm takes as input both the object x *and* the proof that $x \in A$. Hence it cannot in general be rendered a function of x alone [Rat06, §1].

The case of **DC** is different in that we are given an initial object a_1 together with a proof — call it π_1 — that $a_1 \in A$. The premiss then gives us an algorithm that, when given as input an object a_k together with a proof π_k that $a_k \in A$, produces a subsequent object a_{k+1} and proof π_{k+1}. So, given any $n \in \mathbf{N}^+$, we can apply this algorithm $(n - 1)$ times to produce a_n and π_n; thus, we have a_n as a function of n (given (a_1, π_1) as a starting point) [Myh75, pp. 362–363].

$\mathbf{AC}_\omega$, on the other hand, circumvents the problem befalling **AC** by virtue of the fundamental nature of the natural numbers (as well as by the fact that it follows from **DC**): each natural number is, in a sense, its own proof of membership in $\mathbf{N}$. The case for **AC**! is somewhat different: see [Bee85, pp. 42–43] for a proof.

1.6 Bishop-Style Constructive Mathematics

Constructive mathematics underwent a revival with Bishop's 1967 publication of his *Foundations of Constructive Analysis* [Bis67]. In this book, Bishop presented a comprehensive development of analysis using only constructive techniques (but not drawing on any of the additional principles used by Brouwer and the later pioneer Andrey Markov, Jr.). In doing so, he allayed the (at the time) widespread concern that constructive mathematics was too weak to be of any real use.

The school of constructivism that Bishop founded with this work is minimal in the sense that it assumes only that which is common to both Brouwer's intuitionism and Markov's programme of

Russian recursive mathematics;[6] furthermore, it is consistent with classical mathematics. Bishop's framework is now called *Bishop-style constructive mathematics* (**BISH**): we think of it as informal mathematics using intuitionistic logic and assuming the axioms of dependent, countable and unique choice. We then identify the three main "models" of **BISH** as follows.

- Intuitionistic mathematics (**INT**) is **BISH** plus Brouwer's principle of continuous choice[7] and fan theorem.[8]
- Russian recursive mathematics (**RUSS**) is **BISH** plus Markov's principle and the *Church-Markov-Turing thesis*, which asserts that all total functions are recursive.
- Classical mathematics (**CLASS**) is **BISH** plus the full axiom of choice (and hence the law of excluded middle).

It is over the framework of **BISH** that we will work for the majority of this thesis. Why use **BISH**? For one thing, the fact that **INT**, **RUSS** and **CLASS** are extensions of **BISH** means that every proof in **BISH** is also a proof in these other three systems; hence, working in **BISH** gives results that are widely applicable. More important for our purposes, however, is the fact that **BISH** is an ideal common ground over which to analyse the relationships that hold between principles of **INT**, **RUSS** and **CLASS**.

With this in mind, we turn to the project of *constructive reverse mathematics* (CRM), which is concerned with sorting principles from these extensions of **BISH** into equivalence classes relative to **BISH** (or, at least, identifying implication relationships between them). In doing so, we are able to better understand the relationships between a great number of non- and semi-constructive[9] principles, and to identify precisely where the nonconstructivities in many classical results arise from.

Purely constructive methods are unfortunately not enough to reclaim all the theorems of classical mathematics. However, there is not a dichotomy between entirely algorithmic proofs and entirely non-algorithmic proofs: even when a fully constructive proof cannot be found, it is valuable to

[6] Essentially recursive function theory with intuitionistic logic; see [Bee85, Chapter IV].
[7] Refer to [BR87, §5.2].
[8] We will extensively study several variations upon Brouwer's fan theorem. Refer to Section 2.3 for the relevant definitions.
[9] By a *semi-constructive* principle, we mean one which does not hold in **BISH** but *does* hold in some other constructive setting, such as **INT** or **RUSS**. The variations upon Brouwer's fan theorem (introduced in Section 2.3) are examples.

endow at least *some* aspects of the proof with algorithmic content. To form such proofs, we look to extensions of **BISH** found by including some (ideally weak) auxiliary principle. The results of constructive reverse mathematics allow us to establish just how much additional proving power is attained when a particular principle is used in this role, and what non- or semi-constructive role this principle takes. So while not all proofs are created equal, CRM allows us to identify which proofs *are* equal, at least as far as algorithmic content is concerned.

2 | The Road So Far

We now give a more technical introduction to some of the basic concepts that we will use throughout our investigation, and take a brief look at the current state of the constructive reverse mathematics programme. Some familiarity with the elementary theory of metric (and, later, normed) spaces is assumed, as is a basic understanding of intuitionistic logic and the properties of real numbers $\mathbf{R}$ in the constructive setting — refer to [BR87] or [BV06].

We say that a subset $X \subseteq Z$ is *detachable from* Z (or just *detachable*, if the set Z is clear from the context) if, for each $x \in Z$, either $x \in X$ or $x \notin X$. Similarly, a sequence $(z_n)_{n \geqslant 1}$ in Z is X-*detachable* if, for each term z_n, we can decide whether $z_n \in X$ or $z_n \notin X$. A set is *inhabited* if we can construct an element in it. While this is a stronger property than the denial of emptiness, we use the familiar notation $X \neq \varnothing$ to mean that X is inhabited.

Constructively, μ is the (unique) *infimum* of a subset $S \subseteq \mathbf{R}$, written $\mu = \inf S$, if it is a lower bound of S and if, for each $\epsilon > 0$, there exists $x \in S$ such that $x < \mu + \epsilon$. (This definition gives more computational information — and is thus constructively stronger — than simply taking μ to be the greatest lower bound of S [LR10, pp. 6619–6620].) Accordingly, the infimum of a function $f \colon X \to \mathbf{R}$ is given by

$$\inf f \equiv \inf \left\{ f(x) \colon \ x \in X \right\}.$$

(The *suprema* $\sup S$ and $\sup f$ are defined analogously.) Using this notion of infimum, we define the distance from a point z of a metric space Z to a subset $X \subseteq Z$ by

$$\rho(z, X) \equiv \inf \left\{ \rho(z, x) \colon \ x \in X \right\}.$$

If this distance exists for each $z \in Z$, we say that X is *located* in Z.

It is important to note that in the constructive setting, there exist bounded sets that lack infima or suprema. Consider for example the set

$$S \equiv \{0\} \cup \{n \in \mathbf{N} \colon \; n = 1 \wedge P\},$$

where P is some proposition — such as the Goldbach conjecture — whose truth is not known. If we were able to determine a supremum for S, we would, under the BHK interpretation, be able to prove or disprove P. Hence the classical *least-upper-bound principle* — which states that every nonempty set of real numbers that is bounded above has a supremum — does not hold constructively; in fact, it entails **LEM** [BV06, p. 32].

Examining this problem from a constructive perspective, we may improve upon the classical result by identifying precisely what computational information is required in order to find the supremum of a bounded set. We thus obtain the following *constructive least-upper-bound principle*, which is **Theorem 2.1.18** of [BV06, p. 32].

Proposition 2.1: BISH $\vdash$ Let $S \subset \mathbf{R}$ be a set that is bounded above and satisfies the following property: for all rational numbers a, b with $a < b$, either $x \leqslant b$ for all $x \in S$, or else there exists $x \in S$ with $a < x$. Then $\sup S$ exists.

2.1 Specker's Theorem and its Antitheses

The focal principles of this thesis are the so-called *anti-Specker property* and its weakenings. We will be concerned with both

(a) finding implications between these anti-Specker properties and other principles of constructive reverse mathematics, and

(b) exploring how the addition of these principles to **BISH** allows us to produce proofs that are more direct or powerful than would be possible otherwise.

While the value of novel implications is widely acknowledged, direct proofs (of potentially known results) have a more subtle benefit. The crucial thing to note here is that a greater degree of directness makes it easier to see how the *structure* of the relevant objects carries through the proof. Mathematics is — at least in some part — a study of structure: there is thus an immediate (if somewhat abstract) sense in which this directness adds to our mathematical understanding. More concretely, this improved understanding makes it easier to see how the proof can be adapted to other situations, and can lead to greater efficiency when the constructions of the proof are implemented in a computer program.

Why should we care about anti-Specker properties in particular? Firstly, all of the ones that we will examine are provable within **INT** and thus may be considered to be in some sense "semi-constructive." More importantly, though, they allow us to recapture some of the proving power of the inherently nonconstructive property of *sequential compactness* — but more on that shortly.

We will need the following two notions of separation. A sequence (x_n) in a metric space X is said to be:

- *eventually bounded away from the point* $x \in X$ if there exist $N \in \mathbf{N}^+$ and $\delta > 0$ such that $\rho(x, x_n) > \delta$ for all $n \geqslant N$, and
- *eventually bounded away from the subset* $S \subseteq X$ if there exist $N \in \mathbf{N}^+$ and $\delta > 0$ such that $\rho(x, x_n) > \delta$ for all $x \in S$ and $n \geqslant N$.

The anti-Specker properties that we will be interested in now arise from the following famous recursive result of Specker [BR87, p. 58].

Proposition 2.2 (Specker's theorem)**: RUSS** $\vdash$ There exists a nondecreasing sequence of rational numbers in the Cantor set[1] that is eventually bounded away from each point of **R**.

We obtain as a variation upon this theorem the following *Specker property* for the unit interval:

Speck$_{[0,1]}$ There exists a sequence in $[0, 1]$ that is eventually bounded away from each point of $[0, 1]$.

[1]We will make use of the Cantor set C in Chapter 3; until then, it is enough just to recognise here that $C \subset [0, 1]$.

We call such a sequence a *Specker sequence* in $[0,1]$.[2] (Eventually, we will also encounter Specker sequences within certain other metric spaces: these are defined analogously.) From a classical perspective, the existence of such sequences seems markedly counterintuitive: the key thing to recognise is that, within **RUSS**, one deals only with the *computable* real numbers. These Specker sequences are thus realised as sequences of computable real numbers that, classically,[3] converge to uncomputable limits.

As the name suggests, our anti-Specker properties are antitheses of $\mathbf{Speck}_{[0,1]}$, placed within a more general setting. The most well-known of these, first alluded to in [BHS06, p. 734] and later identified in [BB07, pp. 195–196], is the *(full) anti-Specker property* which states, for a subset X of a metric space Z:

$\mathbf{AS}_{X,Z}$ Every sequence in Z that is eventually bounded away from each point of X is eventually bounded away from the entire set X [BB08b, p. 584].

In other words, **AS** allows us to pass from a pointwise characterisation of eventual-bounded-awayness to a uniform one.

What sort of space X does it make sense to apply properties like this to? Clearly, X needs to be in some sense "contained": unbounded spaces like $\mathbf{R}$ yield fairly immediate counterexamples. Furthermore, X should not contain "holes": for, if it did, a sequence in X converging to one of these holes would then provide another counterexample (playing much the same role as our sequences of computable real numbers with uncomputable limits did in the recursive situation). Hence we will require X to have the property of *compactness*, which incorporates these two requirements.

2.2 Compactness

Given $\epsilon > 0$, an ϵ-*approximation* to a metric space X is a subset $Y \subseteq X$ such that for each $x \in X$, there exists $y \in Y$ with $\rho(x, y) < \epsilon$. If for each $\epsilon > 0$ there exists a *finite* ϵ-approximation to X,

[2] Sequences of this kind are often referred to as *strong Specker sequences*, in contrast with a weaker kind of Specker sequence which has only the property of nonconvergence. We will not be concerned with this weaker notion, and thus omit the qualifier "strong."

[3] Under the BHK interpretation, sequences must have a *modulus of convergence* if they are to be considered convergent. In the case of Specker sequences, this modulus is uncomputable.

we say that X is *totally bounded*. Totally bounded spaces have a number of very useful properties: in particular, we will often use the fact that totally bounded subsets of a metric space are located within that space [BV06, p. 41].

Constructively, we consider a *compact* metric space to be one that is both totally bounded and *complete*. But while we adopt this definition for the remainder of this thesis, there are two other widespread characterisations of compactness which, when applied to a metric space X, are classically equivalent to its being complete and totally bounded. Nevertheless, these are of entirely different status when examined within **BISH** (and hence demonstrate how constructive mathematics preserves certain distinctions that classical mathematics cannot).

- *Sequential compactness* asserts that every sequence in X has a convergent subsequence. This notion is very powerful; indeed, too powerful: the claim that every compact metric space is sequentially compact is equivalent to **LPO** [IS04, p. 542]. Hence sequential compactness is, as it stands, irredeemably nonconstructive. However, we can recapture some of the power of sequential compactness, without committing to such a high degree of nonconstructivity, by augmenting **BISH** with an anti-Specker property. **Proposition 3.12** provides a particularly illustrative example.
- *Heine-Borel properties* characterise compactness in terms of open covers: they assert that every open cover of X which satisfies certain properties admits a finite subcover. These are provable in **INT**; indeed, many variations can be established using only a rather weak intuitionistic principle [Die08, §4.2].[4] Consequently, these properties are of some interest in constructive reverse mathematics — see again **Proposition 3.12**. Note, though, that there is a recursive counterexample — now known as the *singular covering theorem* — which renders them false within **RUSS**, and therefore unprovable in **BISH** [BR87, §3.4].

2.3 Fan Theorems

There is another family of principles — the *fan theorems* — that, while secondary to the anti-Specker properties for our purposes, nevertheless play a crucial role in constructive reverse mathematics.

[4] Namely $\mathbf{FT}_\Delta$, which we shall encounter in Section 2.3.

We build up to these with the following preliminary terminology, which originates from [BR87, Chapter 5].

For a set Σ, the set of all finite sequences of elements in Σ (including the empty sequence λ) is denoted Σ^*. If $u \in \Sigma^*$ is such a sequence, we write $|u|$ for the length of u and denote by $\overline{u}n$ the finite sequence consisting of the first n terms of u (extending this latter notation to infinite sequences in the same way). If $v = \overline{u}n$ for some $n \in \mathbf{N}$, we say that v is a *restriction* of u, and that u is an *extension* of v. We write the *concatenation* of u and v as $u * v$, and in the case $v \equiv (x)$ for some $x \in \Sigma$, abbreviate this as $u * x$. Where we have not explicitly named the terms of u, we use the notation $[u]_k$ to denote the k^{th} term.

A *fan* Γ is a detachable subset of $\mathbf{N}^*$ that is closed under restriction, and is such that for each $u \in \Gamma$, the set $\{n \in \mathbf{N}: \ u * n \in \Gamma\}$ is inhabited and finite.[5] In particular, we will be interested in the *complete binary fan* $\mathbf{2}^*$, which corresponds to the set of all finite binary sequences.

A *path* in a fan Γ is a sequence α — which may be finite or infinite — such that $\overline{\alpha}n \in \Gamma$ for each applicable $n \in \mathbf{N}$. Hence paths in $\mathbf{2}^*$ are just (finite or infinite) binary sequences. We say that a path α is *blocked* by a subset $B \subseteq \mathbf{2}^*$ if some restriction of α belongs to B. Alternatively, if no restriction of α is in B, we say that α *misses* B.

The fan theorems concern the relationship between two special types of subset of $\mathbf{2}^*$: bars and uniform bars. A subset $B \subseteq \Gamma$ is a *bar* (for Γ) if each infinite path of Γ is blocked by B. Furthermore, B is a *uniform bar* if there exists a positive integer N such that each finite path of length N is blocked by B. Clearly, every uniform bar is a bar.

Suppose that ? denotes a property of subsets of $\mathbf{2}^*$. Then Brouwer's *fan theorem for ?-bars*, $\mathbf{FT}_?$, is the statement that every bar for $\mathbf{2}^*$ with the property ? is a uniform bar. What sort of property might this be? The strongest that we will consider is detachability; the weakest is when we require nothing at all of the bar in question. Accordingly, the weakest fan theorem of this form that we will consider is the *fan theorem for detachable bars*:

$\mathbf{FT}_\Delta$ Every detachable bar for $\mathbf{2}^*$ is uniform.

[5] In Brouwer's terminology, a fan is a *finitely-branching spread* [Lie04, p. 19].

And the strongest is the *full fan theorem*:

> $\mathbf{FT}_{\mathrm{Full}}$ Every bar for $\mathbf{2}^*$ is uniform.

However, there are several interesting variations upon the fan theorem that lie (strictly![6]) between these two extremes. To describe these, we identify a further three different types of subset of $\mathbf{2}^*$. A *c-subset* is a set $B \subseteq \mathbf{2}^*$ such that

$$B = \Big\{ u \in \mathbf{2}^* \colon \; (\forall v \in \mathbf{2}^*)\big[u * v \in D)\big] \Big\},$$

for some detachable subset $D \subseteq \mathbf{2}^*$. Intuitively, one obtains a c-subset from a detachable subset D by discarding the parts of D that are not closed under extension. This type of set was introduced by Berger in [Ber06, pp. 36–37]: the name "c-set" comes from the fact that the *fan theorem for c-bars* $\mathbf{FT}_c$ is equivalent to a continuity property for functions from $\mathbf{2}^{\mathbf{N}^+}$ to $\mathbf{N}^+$.

Although it is not necessarily the case that a detachable bar is a c-bar, it is straightforward to see that $\mathbf{FT}_c$ entails $\mathbf{FT}_\Delta$: assume $\mathbf{FT}_c$ and, given a detachable bar B, let

$$B' \equiv \Big\{ u \colon \; (\exists n)\big[\overline{u}n \in B\big] \Big\}$$

be the detachable bar obtained by closing B under extensions. Then

$$u \in B' \iff (\forall v)\big[u * v \in B'\big],$$

so B' is a c-bar and therefore uniform. It follows that B must be uniform also.

Another type of subset of $\mathbf{2}^*$ — the Π^0_1*-subset* — is defined throughout the relevant literature in one of two seemingly distinct ways:[7]

- A subset of the form

$$B = \Big\{ u \in \mathbf{2}^* \colon \; (\forall n \in \mathbf{N})\big[(u, n) \in D\big] \Big\},$$

[6] See [DL13].

[7] Publications referring to Π^0_1-subsets in the *weaker* (Π^0_1) sense include [Ber09a], [Ber09b], [Ber06], [DS10] and [DL13]; publications referring to Π^0_1-subsets in the *stronger* (Π^0_1cl) sense include [BB06], [Bri08], [DL09] and [Die08].

for some detachable set $D \subseteq \mathbf{2}^* \times \mathbf{N}$. We call sets meeting this condition Π^0_1-subsets because, for each finite path u, the assertion $u \in B$ is classified as a Π^0_1 formula in the arithmetical hierarchy of recursion theory [Bee85, p. XXI].

❖ A subset of the same form as the one above, but with the following additional requirement on the set D:

$$(\forall u \in \mathbf{2}^*)(\forall n \in \mathbf{N})\big[(u,n) \in D \implies (u * 0, n) \in D \wedge (u * 1, n) \in D\big].$$

We call sets meeting this condition Π^0_1cl-subsets (where "cl" stands for "closed"), so as to avoid ambiguity with the previous definition.

Clearly, all Π^0_1cl-subsets are Π^0_1-subsets; furthermore, it is known that all c-subsets are Π^0_1cl-subsets. Hence we have the following implications:

$$\mathbf{FT}_{\text{Full}} \implies \mathbf{FT}_{\Pi^0_1} \implies \mathbf{FT}_{\Pi^0_1\text{cl}} \implies \mathbf{FT}_{\text{c}} \implies \mathbf{FT}_{\Delta}.$$

An investigation of Diener and Lubarsky [DL13] has demonstrated that the implications from $\mathbf{FT}_{\text{c}}$ to $\mathbf{FT}_{\Delta}$, from $\mathbf{FT}_{\Pi^0_1}$ to $\mathbf{FT}_{\text{c}}$, and from $\mathbf{FT}_{\text{Full}}$ to $\mathbf{FT}_{\Pi^0_1}$ are strict. It would also seem that despite the lack of distinction between $\mathbf{FT}_{\Pi^0_1}$ and $\mathbf{FT}_{\Pi^0_1\text{cl}}$ in the literature, there is no evidence to suggest that these should be equivalent.

Why are we interested in these principles? $\mathbf{FT}_{\text{Full}}$ is a cornerstone of Brouwer's intuitionism: it, and therefore all of the weaker fan theorems, are true within **INT**. But more importantly for our purposes, they provide a neat hierarchy to which other principles may be linked and thus organised, in the spirit of constructive reverse mathematics. In particular, the version of the full anti-Specker property that we will be most interested in[8] has been shown by Berger and Bridges [BB07] to be equivalent to $\mathbf{FT}_{\text{c}}$, and the *limited anti-Specker property*, which we shall encounter in Chapter 3, seems to fall somewhere between $\mathbf{FT}_{\text{c}}$ and $\mathbf{FT}_{\Delta}$.

[8]See Section 3.1.

2.4 Continuity

We conclude this chapter with an exposition of one of the most important concepts underpinning our subsequent explorations: *continuity*. Let f be a mapping from a metric space X into a metric space Y. We say that f is, in (roughly) increasing order of strength:

- *sequentially continuous* if for each $x \in X$, we have $f(x_n) \to f(x)$ whenever (x_n) is a sequence in X with $x_n \to x$;

$$(\forall x \in X)(\forall (x_n) \in X^{\mathbf{N}^+})\left[\lim_{n\to\infty} \rho(x_n, x) = 0 \implies \lim_{n\to\infty} \rho\big(f(x_n), f(x)\big) = 0\right]$$

- *(pointwise) continuous* if for each $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(f(x), f(x')) < \epsilon$ whenever $x' \in X$ is a point with $\rho(x, x') < \delta$;

$$(\forall x \in X)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x' \in X)\left[\rho(x, x') < \delta \implies \rho\big(f(x), f(x')\big) < \epsilon\right]$$

- *uniformly sequentially continuous* if $\rho(f(x_n), f(x'_n)) \to 0$ whenever (x_n) and (x'_n) are sequences in X with $\rho(x_n, x'_n) \to 0$; and

$$(\forall (x_n), (x'_n) \in X^{\mathbf{N}^+})\left[\lim_{n\to\infty} \rho(x_n, x'_n) = 0 \implies \lim_{n\to\infty} \rho\big(f(x_n), f(x'_n)\big) = 0\right]$$

- *uniformly continuous* if for each $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(f(x), f(x')) < \epsilon$ whenever $x, x' \in X$ are points with $\rho(x, x') < \delta$.

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, x' \in X)\left[\rho(x, x') < \delta \implies \rho\big(f(x), f(x')\big) < \epsilon\right]$$

One property of uniformly continuous functions that we will frequently rely on is **Corollary 2.2.7** of [BV06, p. 40]: if f is a uniformly continuous function from a totally bounded space into $\mathbf{R}$, then $\inf f$ and $\sup f$ both exist (though f need not attain these!).

Relative to **BISH**, uniform continuity implies pointwise and uniform sequential continuity, which both imply sequential continuity. However, any implications beyond these correspond to principles not provable within **BISH** alone [Bri09b, p. 438]. We identify two in particular. The *uniform continuity theorem* for metric spaces X and Y states:

UCT$_{X,Y}$ Every pointwise continuous function $f\colon X \to Y$ is uniformly continuous.

Similarly, we have the *uniform sequential continuity theorem*:

USCT$_{X,Y}$ Every pointwise continuous function $f\colon X \to Y$ is uniformly sequentially continuous.

While we will not have cause to study these two properties directly, they nevertheless warrant mention due to their significance in the CRM programme. **UCT**$_{[0,1],\mathbf{R}}$ in particular is striking in that it falls neatly[9] into the hierarchy of fan theorems that serves as a backbone for a significant portion of constructive reverse mathematics, as depicted in **Figure 2.2**. Furthermore, our introduction of these principles here sets the stage for Chapter 4, in which we will encounter their antitheses **anti-UCT** and **anti-USCT**.

Another principle indispensable to continuity relationships in the CRM literature is *Ishihara's (boundedness) principle* **BD-N**, first introduced in [Ish92, p. 561]. We say that a subset $S \subseteq \mathbf{N}$ is *pseudobounded* if, for each sequence $(s_n)_{n\geqslant 1}$ in S, we have $\lim_{n\to\infty} \frac{s_n}{n} = 0$ (or equivalently $s_n < n$ eventually, due to Lemma 3 of [IY02, p. 1515]). **BD-N** then asserts:

Every countable inhabited pseudobounded subset of **N** is bounded.

Note that the requirement of countability here is not redundant: constructively, it is not the case that every subset of **N** is countable (in the sense that there exist a detachable subset $D \subseteq \mathbf{N}^+$ and a mapping of D onto S). As an example, consider again the set $S \equiv \{0\} \cup \{n \in \mathbf{N} \colon\ n = 1 \wedge P\}$, where the truth of P is not known.

For functions from $[0, 1]$ to **R**, Ishihara's principle allows one to pass from sequential continuity to pointwise continuity, or from uniform sequential continuity to uniform continuity, as depicted in **Figure 2.1**.[10] (In fact, **BD-N** is *equivalent* to these continuity relationships for functions from a

[9] That is, between **FT**$_{\Pi^0_1\text{cl}}$ [DL09, p. 52] and **FT**$_c$ [Ber06, p. 38].

[10] The one unfamiliar principle in this diagram, **SPOS**, is the *strong positivity property* from Chapter 4.

separable metric space into a metric space.) It is also significant in that it holds in all three major models of **BISH** (that is: **INT**, **RUSS** and **CLASS**) [Bri08, p. 324]; however, it is not provable within **BISH** alone [LS12].

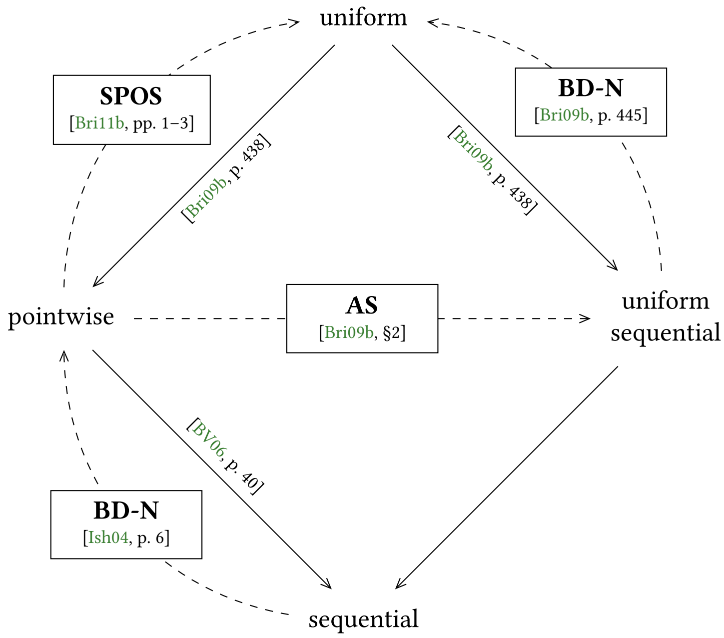


Figure 2.1: Continuity relationships for functions from $[0, 1]$ to $\mathbf{R}$

Figure 2.2 gives an overview of the current state of the CRM programme. Many of the principles it incorporates are beyond the scope of our exposition here; many we will not encounter until later chapters: the reader should refer to the indicated references for more details.

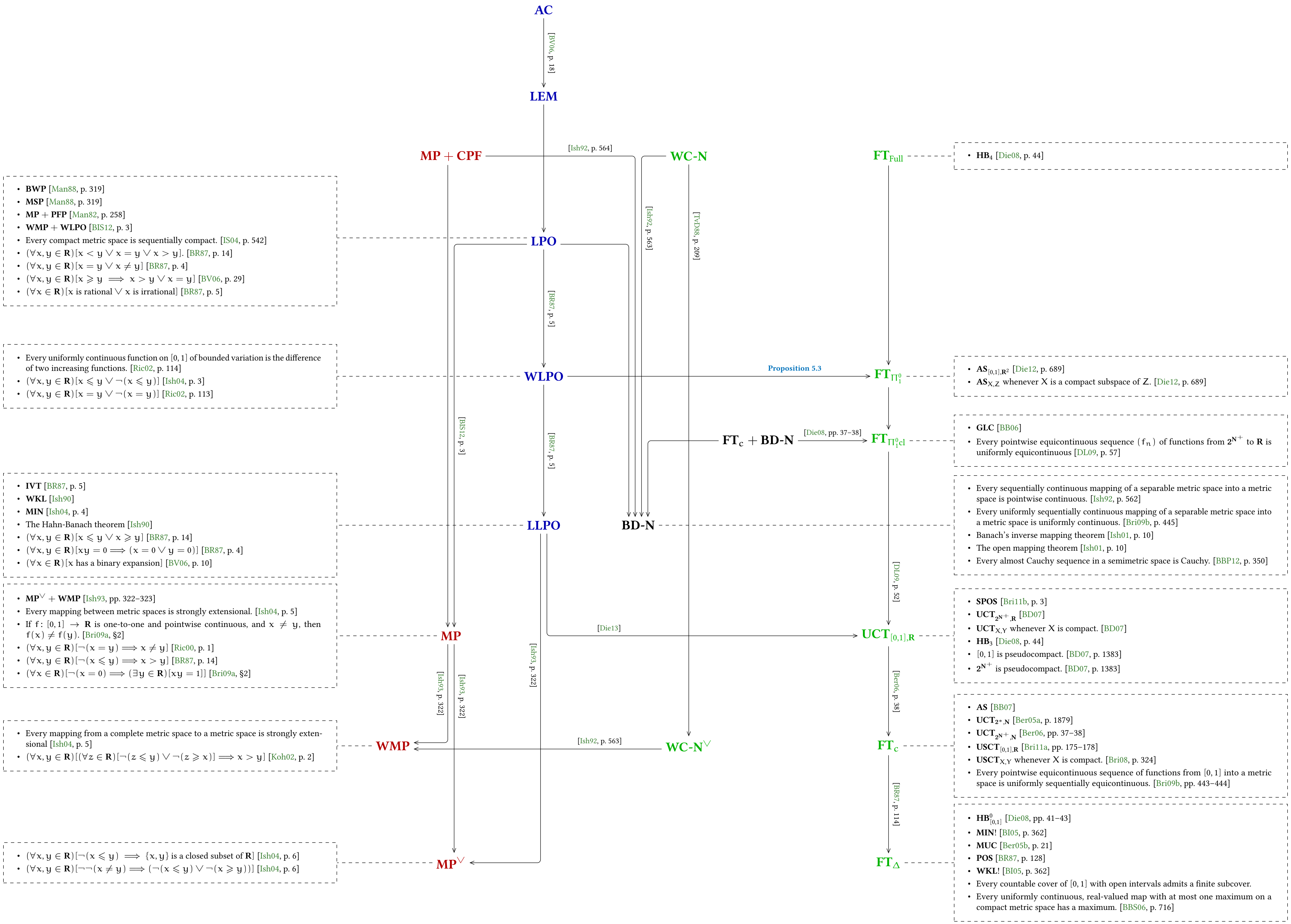


Figure 2.2: An overview of the current state of the CRM programme

3 | Weak Anti-Specker Properties

3.1 Anti-Specker, Again

Recall from Section 2.1 the *anti-Specker property*, which states, for a (compact) subset X of a metric space Z:

AS$_{X,Z}$ If $(z_n)_{n \geqslant 1}$ is a sequence in Z that is eventually bounded away from each point of X, then (z_n) is eventually bounded away from the entire set X.

In this chapter, we introduce and examine some natural weakenings of this principle. First, though, some further classification of this "full" anti-Specker property is in order.

While the situation of **AS**$_{X,Z}$ within the greater CRM picture depends upon the choice of the spaces X and Z, it turns out that the variations we will be most interested in fall into a single equivalence class. In particular, we shall be concerned with the case where $X = [0, 1]$ and $Z = [0, 1] \cup \{2\}$ (or equivalently, $Z = \mathbf{R}$ [Bri09b, pp. 439–440]) and, more generally, the case where X is a compact space and $Z = X \cup \{\xi\}$ is some one-point extension of X (so that $\rho(\xi, X) > 0$): in such a setting, being bounded away from X means being equal to the added point. We write **AS**$_X$ as shorthand for the case where Z is such a one-point extension of X (or, equivalently, *any* such one-point extension [BD10, p. 435]).[1] This brings us to the following result.

[1]In the terminology of [BD10, p. 435], this is the *unrelativised* anti-Specker property for X.

Proposition 3.1: **BISH** $\vdash$ The following are equivalent.

(i) $\mathbf{AS}_{[0,1]}$.

(ii) $\mathbf{AS}_{\mathbf{2}^{\mathbf{N}^+}}$.

(iii) $\mathbf{AS}_X$ for each inhabited compact space X.

In light of this equivalence, we denote these properties ((iii) in particular) by simply **AS**, where convenient.

$\mathbf{2}^{\mathbf{N}^+}$ is the *Cantor space* of infinite binary sequences, with the metric

$$\rho(\alpha, \beta) = \inf\left\{2^{-k}: \ \overline{\alpha}k = \overline{\beta}k\right\}.$$

Its role here is to facilitate the passage from (i) to (iii): every inhabited compact space X is a so-called *uniform quotient* of $\mathbf{2}^{\mathbf{N}^+}$ [BR87, p. 106], and this fact allows us to construct a uniformly continuous mapping of $\mathbf{2}^{\mathbf{N}^+}$ onto X. We will encounter several results like **Proposition 3.1** throughout this chapter, all of which exploit this property.

The relationship between $\mathbf{AS}_{[0,1]}$ and $\mathbf{AS}_{\mathbf{2}^{\mathbf{N}^+}}$ has been indirectly established by Berger and Bridges in [BB07, pp. 199–200] and Diener in [Die08, pp. 36–37], who have shown that $\mathbf{AS}_{[0,1]} \implies \mathbf{FT}_c$ and $\mathbf{FT}_c \implies \mathbf{AS}_{\mathbf{2}^{\mathbf{N}^+}}$, respectively. Combining and distilling their proofs, we obtain the following construction. As in [BB08a, p. 133], define an embedding $F\colon \mathbf{2}^{\mathbf{N}^+} \cup \mathbf{2}^* \to [0,1]$ of paths in $\mathbf{2}^*$ into the unit interval by

$$F(\alpha) = \sum_{k=1}^{|\alpha|} [\alpha]_k 2^{-k},$$

where we write ∞ for $|\alpha|$ when $\alpha \in \mathbf{2}^{\mathbf{N}^+}$. Now fix $(u_n)_{n\geqslant 1}$ to be the intuitive one-one enumeration $(\lambda, 0, 1, 00, 01, 10, 11, 000, \dots)$ of $\mathbf{2}^*$. If $(\alpha_n)_{n\geqslant 1}$ is a sequence in a one-point extension $\mathbf{2}^{\mathbf{N}^+} \cup \{\omega\}$ of the Cantor space that is eventually bounded away from each point of $\mathbf{2}^{\mathbf{N}^+}$, then the sequence $(x_n)_{n\geqslant 1}$ in $[0,1] \cup \{2\}$ defined by

$$x_n = \begin{cases} F(u_n) & \text{if } u_n \text{ is a restriction of } \alpha_{|u_n|}, \text{ and} \\ 2 & \text{otherwise} \end{cases}$$

is eventually bounded away from each point of $[0,1]$ [BB07, p. 200]. Furthermore, if $x_n = 2$ eventually, it follows that $\alpha_n = \omega$ eventually; hence this construction shows how the anti-Specker property in $[0,1]$ carries over into $\mathbf{2}^{\mathbf{N}^+}$.

However, this approach does not adapt well to prove similar results about the weaker anti-Specker properties we will shortly consider.[2] Fortunately, we can make a construction that more faithfully carries the structure of $\mathbf{2}^{\mathbf{N}^+}$ over into $[0,1]$. Note, though, that our embedding F of paths in $\mathbf{2}^{\mathbf{N}^+}$ into $[0,1]$ will not serve here, as points that are near each other in $[0,1]$ may (under F) correspond to points potentially very far apart in $\mathbf{2}^{\mathbf{N}^+}$. (For example, the paths $\alpha = 0111\ldots$ and $\beta = 1000\ldots$ have $|F(\alpha) - F(\beta)| = 0$ but $\rho(\alpha, \beta) = 1$!) To get around this problem, we instead use the embedding

$$G(\alpha) \equiv \sum_{k=1}^{|\alpha|} \frac{2[\alpha]_k}{3^{k+1}}$$

of each $\alpha \in \mathbf{2}^{\mathbf{N}^+}$ into the Cantor set $C \equiv G\left(\mathbf{2}^{\mathbf{N}^+}\right) \subset [0,1]$.

Proposition 3.2: BISH $\vdash$ Let $(\alpha_n)_{n\geqslant 1}$ be a sequence in $\mathbf{2}^{\mathbf{N}^+} \cup \{\omega\}$ that is eventually bounded away from each point of $\mathbf{2}^{\mathbf{N}^+}$. Then there exists a sequence $(x_n)_{n\geqslant 1}$ in $[0,1] \cup \{2\}$ that is eventually bounded away from each point of $[0,1]$, and has $x_n = 2$ when and only when $\alpha_n = \omega$.

Proof. Define $(x_n)_{n\geqslant 1}$ by

$$x_n = \begin{cases} G(\alpha_n) & \text{if } \alpha_n \in \mathbf{2}^{\mathbf{N}^+}, \text{ and} \\ 2 & \text{if } \alpha_n = \omega \end{cases}$$

for each n: we show that (x_n) is eventually bounded away from each point of $[0,1]$. Fix any such point x. The set C is complete and located in $[0,1]$ [BR87, p. 58]; hence, we can apply *Bishop's lemma* (**Proposition 3.1.1** of [BV06, p. 64]) to find a point $c \in C$ such that if $x \neq c$, then $\rho(x, C) > 0$.

Since $c \in C$, we can find $\alpha \in \mathbf{2}^{\mathbf{N}^+}$ such that $c = G(\alpha)$. But (α_n) is eventually bounded away from α, so there exist $\delta > 0$ and $M \in \mathbf{N}^+$ such that

$$\rho(\alpha_n, \alpha) \equiv \inf\left\{2^{-k} :\ \overline{\alpha_n}(k) \neq \overline{\alpha}(k)\right\} > \delta$$

[2] In fact, while it seems that this construction is of little use in the case for the *limited anti-Specker property* (page 27), it can be adapted to prove that the *non-Specker property* for $[0,1]$ carries over into $\mathbf{2}^{\mathbf{N}^+}$ — see page 36.

for all $n \geqslant M$ with $\alpha_n \in \mathbf{2}^{\mathbf{N}^+}$. Hence we can find an index $N \in \mathbf{N}^+$ such that $\overline{\alpha_n}(N) \neq \overline{\alpha}(N)$ (that is, $[\alpha_n]_\ell \neq [\alpha]_\ell$ for some $\ell \leqslant N$) for all $n \geqslant N$ with $\alpha_n \in \mathbf{2}^{\mathbf{N}^+}$. Then for each such n,

$$\left|x_n - c\right| = \left|G(\alpha_n) - G(\alpha)\right| \geqslant 3^{-\ell} \geqslant 3^{-N} \text{ [BR87, p. 58].}$$

Two cases now arise:

- If $x \neq c$, then $\rho(x, C) > 0$, and since (x_n) is contained within $C \cup \{2\}$, it follows that $|x_n - x| \geqslant \rho(x, C) > 0$ for all n.
- If $|x - c| < 2 \cdot 3^{-N-1}$, then for all $n \geqslant N$ with $x_n \in [0, 1]$ we have

$$\left|x_n - x\right| \geqslant \left|x_n - c\right| - \left|c - x\right| > 3^{-N} - 2 \cdot 3^{-N-1} = 3^{-N-1}.$$

Since $|x_n - x| \geqslant 1$ when $x_n = 2$, we see that, in either of these cases, (x_n) is eventually bounded away from x. □

This brings us back to our original classification result.

Proof of Proposition 3.1. (iii) $\Longrightarrow$ (i) is trivial, and (i) $\Longrightarrow$ (ii) follows from **Proposition 3.2**. To prove that (ii) $\Longrightarrow$ (iii), let X be a compact metric space with one-point extension $X \cup \{\xi\}$ (where $\rho(\xi, X) > 0$). **Theorem 1.4** of [BR87, p. 106] allows us to find a continuous mapping of $\mathbf{2}^{\mathbf{N}^+}$ onto X, and **Proposition 10** of [BD10, p. 438] now gives the desired result. □

3.2 The Limited Anti-Specker Property

We now introduce our first weak anti-Specker property: the *limited anti-Specker property*. For a (compact) subset X of a metric space Z, this states:

$\mathbf{AS}^{\mathbf{ltd}}_{X,Z}$ If $(z_n)_{n \geqslant 1}$ is a sequence in Z that is eventually bounded away from each point of X, then there exists $k \in \mathbf{N}^+$ such that $\rho(z_k, X) > 0$.

Notice that by a simple induction argument, $\mathbf{AS}^{\mathbf{ltd}}_{X,Z}$ is equivalent to the following claim:

> If $(z_n)_{n\geqslant 1}$ is a sequence in Z that is eventually bounded away from each point of X, then there exists a subsequence $(z_{n_k})_{k\geqslant 1}$ such that $\rho(z_{n_k}, X) > 0$ for each k.

As with the full anti-Specker property, we write $\mathbf{AS}^{\mathbf{ltd}}_{X}$ when the space Z is simply a one-point extension of X, and have the following classification result, in accordance with which we simply write $\mathbf{AS}^{\mathbf{ltd}}$ to refer to these properties (formulation (iii) in particular).

Proposition 3.3: BISH $\vdash$ The following are equivalent.

(i) $\mathbf{AS}^{\mathbf{ltd}}_{[0,1]}$.

(ii) $\mathbf{AS}^{\mathbf{ltd}}_{2^{\mathbf{N}^+}}$.

(iii) $\mathbf{AS}^{\mathbf{ltd}}_{X}$ for each inhabited compact space X.

Proof. The argument used to prove **Proposition 3.1** via **Proposition 3.2** can be adapted, with at most minor modification, to apply here. (It would seem that the more circuitous construction of page 25 will not do.) □

It seems as though $\mathbf{AS}^{\mathbf{ltd}}$ ought to be strictly weaker than **AS**. To determine just *how much* weaker it is, we search for principles whose addition to **BISH** would allow one to pass from the former to the latter. Our next two results show that, in the presence of **MP**, $\mathbf{AS}^{\mathbf{ltd}}$ is in some sense "close" to **AS** [BDMJ12].

Proposition 3.4: BISH + MP + $\mathbf{AS}^{\mathbf{ltd}}$ $\vdash$ If $(z_k)_{k\geqslant 1}$ is a sequence in $[0,1]\cup\{2\}$ that is eventually bounded away from each point of $[0,1]$, and $(n_k)_{k\geqslant 1}$ is a strictly increasing sequence of positive integers, then there exists $K \in \mathbf{N}^+$ such that $z_j = 2$ whenever $n_K < j < n_{K+1}$.

Proof. Having fixed appropriate sequences (z_n) and (n_k), construct a binary sequence $(\lambda_k)_{k\geqslant 1}$ such that

$$\lambda_k = 0 \implies (\exists j\colon n_k < j < n_{k+1})\big[z_j \in [0,1]\big], \text{ and}$$
$$\lambda_k = 1 \implies (\forall j\colon n_k < j < n_{k+1})\big[z_j = 2\big].$$

Suppose that $\lambda_k = 0$ for all k. Then we obtain a strictly increasing sequence of positive integers $(j_k)_{k \geqslant 1}$ such that $z_{j_k} \in [0, 1]$ for all k. But since (z_{j_k}) is a subsequence of (z_k), it must be eventually bounded away from each point of X, whence by $\mathbf{AS}^{\mathbf{ltd}}$ we have $z_{j_K} = 2$ for some K: a contradiction. Hence it is impossible that $\lambda_k = 0$ for all k, and we conclude by **MP** that there exists K for which $\lambda_K = 1$: that is, $z_j = 2$ for all j with $n_K < j < n_{K+1}$. □

Corollary 3.5: $\mathbf{BISH} + \mathbf{MP} + \mathbf{AS}^{\mathbf{ltd}} \vdash$ If $(z_k)_{k \geqslant 1}$ is a sequence in $[0, 1] \cup \{2\}$ that is eventually bounded away from each point of $[0, 1]$, then for each ℓ and M in $\mathbf{N}^+$, there exists $m \geqslant M$ such that $z_j = 2$ whenever $m \leqslant j < m + \ell$.

Proof. Define a strictly increasing sequence of positive integers $(n_k)_{k \geqslant 1}$ by setting $n_1 = M$ and taking $n_{k+1} = n_k + \ell + 1$ for each $k \in \mathbf{N}^+$. Then by Proposition 3.4, there exists $K \in \mathbf{N}^+$ such that $z_j = 2$ whenever $n_K < j < n_{K+1}$. Taking $m = n_K + 1$ now gives the desired result. □

Thus, in the presence of **MP**: given a sequence in $[0, 1] \cup \{2\}$ that is eventually bounded away from each point of $[0, 1]$, $\mathbf{AS}^{\mathbf{ltd}}$ enables us to find arbitrarily long stretches of terms of (z_n) that are equal to 2, arbitrarily far out in the sequence.

Notice that this proof does not *directly* utilise the fact that the sequence (z_n) is eventually bounded away from each point of $[0, 1]$; rather, it proceeds only from the structure of its terms at 2. Ideally, we would like to make use of this special property of (z_n) when attempting to close the gap between **AS** and $\mathbf{AS}^{\mathbf{ltd}}$; otherwise, we run the risk of helping ourselves to more proving power than we actually need.

Another principle that may be of some utility here is **BD-N**, which seems as though it could be used to facilitate the following kind of argument. Start by fixing a sequence $(z_n)_{n \geqslant 1}$ in $[0, 1] \cup \{2\}$ that is eventually bounded away from each point of $[0, 1]$, and consider its index set $N_0 = \mathbf{N}^+$. The limited anti-Specker property allows us to find a strictly increasing sequence $(n_k)_{k \geqslant 1}$ in N_0 such that $z_{n_k} = 2$ for all k; having constructed this sequence, consider the new index set $N_1 = N_0 \setminus \{n_k : k \in \mathbf{N}^+\}$. Repeating this process, we continue to remove countable families of elements from the index set. If we could somehow use **BD-N** to show that this eventually produces a bounded index set N_i, it would follow that $z_n = 2$ eventually, thereby establishing **AS**.

Although $\mathbf{AS}^{\mathbf{ltd}}$ appears to be weaker than **AS**, there are many situations in which it suffices to prove important results of analysis. We now give two such applications.

3.2.1 $\mathbf{AS}^{\mathrm{ltd}}$ and Positivity

Positivity is a property that has been studied in modern constructive reverse mathematics since its genesis, which is generally considered to be marked by a paper of Julian and Richman featuring the following result [JR84, pp. 337–338].

Proposition 3.6: **BISH** $\vdash$ Given a nonnegative uniformly continuous function $f\colon [0,1] \to \mathbf{R}$, we can construct a detachable subset $B \subseteq \mathbf{2}^*$ such that

(i) $f(x) > 0$ for all $x \in [0,1]$ if and only if B is a bar, and

(ii) $\inf f > 0$ if and only if B is a uniform bar.

Conversely, given a detachable $B \subseteq \mathbf{2}^*$, we can construct a nonnegative uniformly continuous function $f\colon [0,1] \to \mathbf{R}$ that satisfies (i) and (ii).[3]

It is a consequence of this theorem that $\mathbf{FT}_\Delta$ is equivalent to the *positivity property*, which states, for a (compact) space X:

$\mathbf{POS}_X$ Each uniformly continuous function $f\colon X \to \mathbf{R}^+$ has positive infimum.

This leads to another classification result in the same vein as **Proposition 3.1** and **Proposition 3.3**, in light of which we simply write **POS** for the following properties.

Proposition 3.7: **BISH** $\vdash$ The following are equivalent.

(i) $\mathbf{POS}_{[0,1]}$.

(ii) $\mathbf{POS}_{\mathbf{2}^{\mathbf{N}^+}}$.

(iii) $\mathbf{POS}_X$ for each inhabited compact space X.

[3] The original formulation of this theorem was in terms of a detachable subset $F \subseteq \mathbf{2}^*$ closed under restrictions. We here take $B \equiv \mathbf{2}^* \setminus F$.

Proof. (iii) $\implies$ (i) is trivial. For (i) $\implies$ (ii), recall that $\mathbf{POS}_{[0,1]}$ entails $\mathbf{FT}_\Delta$, by the Julian-Richman result (**Proposition 3.6**). An argument of Berger and Ishihara [BI05, p. 362] then shows that $\mathbf{FT}_\Delta$ in turn entails $\mathbf{POS}_{\mathbf{2}^{\mathbf{N}^+}}$.

To prove that (ii) $\implies$ (iii), fix any compact metric space X and uniformly continuous function $f\colon X \to \mathbf{R}^+$. As in the proof of **Proposition 3.1**, **Theorem 1.4** of [BR87, p. 106] allows us to find a uniformly continuous, surjective mapping $g\colon \mathbf{2}^{\mathbf{N}^+} \to X$. Then by the surjectivity of g,

$$\inf f \equiv \inf\left\{f(x)\colon\ x \in X\right\} = \inf\left\{f(g(\alpha))\colon\ \alpha \in \mathbf{2}^{\mathbf{N}^+}\right\} \equiv \inf f \circ g.$$

Moreover, the composite function $f \circ g\colon X \to \mathbf{R}^+$ is uniformly continuous; hence we can apply $\mathbf{POS}_X$ to obtain $\inf f = \inf f \circ g > 0$. ☐

Having appropriately identified **POS**, we now explore its relationship with $\mathbf{AS}^{\mathbf{ltd}}$. In particular, we can prove the following [BDMJ12].

Proposition 3.8: $\mathbf{BISH} + \mathbf{AS}^{\mathbf{ltd}} \vdash$ If $f\colon [0,1] \to \mathbf{R}^+$ is pointwise continuous and has infimum μ, then $\mu > 0$.

Proof. Let $f\colon [0,1] \to \mathbf{R}^+$ be a pointwise continuous function with infimum μ. Define a nondecreasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \implies \mu < 2^{-n}, \text{ and}$$
$$\lambda_n = 1 \implies \mu > 2^{-n-1}.$$

Now construct a sequence $(z_n)_{n\geqslant 1}$ in $[0,1] \cup \{2\}$ as follows:

- if $\lambda_n = 0$, choose $z_n \in [0,1]$ so that $f(z_n) < 2^{-n}$;
- if $\lambda_n = 1$, set $z_n = 2$.

We show that (z_n) is eventually bounded away from each point of $[0,1]$: fix $x \in [0,1]$. Pick a positive integer N such that $f(x) > 2^{-N}$. The continuity of f at x allows us to compute $\delta \in (0,1)$

such that if $x' \in [0,1]$ is a point with $|x - x'| < \delta$, then $|f(x) - f(x')| < 2^{-N-1}$. Now consider any $n > N$: if $\lambda_n = 0$, then

$$f(x) - f(z_n) > 2^{-N} - 2^{-n} \geqslant 2^{-N} - 2^{-N-1} = 2^{-N-1}$$

and therefore $|z_n - x| \geqslant \delta$; if $\lambda_n = 1$, then this inequality holds immediately. Hence (z_n) is eventually bounded away from x, and we can apply $\mathbf{AS}^{\mathbf{ltd}}_{[0,1]}$ to find $k \in \mathbf{N}^+$ for which $z_k = 2$ and therefore $\lambda_k = 1$, whence $\mu > 2^{-k-1} > 0$. □

Corollary 3.9: $\mathbf{BISH} + \mathbf{AS}^{\mathbf{ltd}} \vdash \mathbf{POS}$.

Proof. Suppose that $f\colon [0,1] \to \mathbf{R}^+$ is uniformly continuous. By **Corollary 2.2.7** of [BV06, p. 40], the infimum of f exists. Hence we can apply **Proposition 3.8** to deduce that $\inf f > 0$. □

This relationship has further significance in that it allows us to connect $\mathbf{AS}^{\mathbf{ltd}}$ to the hierarchy of fan theorems, via the Julian-Richman result (**Proposition 3.6**).

Corollary 3.10: $\mathbf{BISH} + \mathbf{AS}^{\mathbf{ltd}} \vdash \mathbf{FT}_\Delta$.

It would now be very desirable to determine whether or not this implication can be reversed. If such a reversal were possible, we would have a rather satisfying fan-theoretic equivalent (namely $\mathbf{FT}_\Delta$) for the limited anti-Specker property; if not, we would be able to place $\mathbf{AS}^{\mathbf{ltd}}$ somewhere between $\mathbf{FT}_c$ and $\mathbf{FT}_\Delta$ — provided, that is, that $\mathbf{AS}^{\mathbf{ltd}}$ is indeed strictly weaker than **AS**.[4]

3.2.2 $\mathbf{AS}^{\mathbf{ltd}}$ and a Heine-Borel Property

For our next application, we will see how $\mathbf{AS}^{\mathbf{ltd}}$ allows us to establish the *countable Heine-Borel property for intervals*, which states, for a (compact) subset $K \subset \mathbf{R}$:

$\mathbf{HB}^0_K$ If $(I_n)_{n \geqslant 1}$ is a sequence of inhabited, bounded open intervals such that $K \subseteq \bigcup_{i=1}^{\infty} I_i$, then there exists $k \in \mathbf{N}^+$ such that $K \subseteq \bigcup_{i=1}^{k} I_i$.

[4]Recall from Section 2.3 that **AS** is equivalent to $\mathbf{FT}_c$ over **BISH** [BB07].

This principle characterises compactness in terms of open covers (recall Section 2.2) and can, in the classical setting, be easily proved using sequential compactness.

Proposition 3.11: CLASS $\vdash$ $\mathbf{HB}^0_K$ for each compact $K \subset \mathbf{R}$.

Proof. Fix a compact set $K \subset \mathbf{R}$, and let $(I_n)_{n\geqslant 1}$ be a sequence of nonempty, bounded open intervals such that $K \subseteq \bigcup_{i=1}^{\infty} I_i$. Suppose that, for all $n \in \mathbf{N}^+$, we have $K \nsubseteq \bigcup_{i=1}^{n} I_i$. Then for each such n, we can choose $z_n \in K$ such that $z_n \notin \bigcup_{i=1}^{n}$. Since K is compact, it is sequentially compact; this allows us to extract a convergent subsequence $(z_{n_k})_{k\geqslant 1}$ from the sequence thus defined.

Let $z \in K$ be the limit of this subsequence, and find an index ℓ such that $z \in I_\ell$. Since I_ℓ is open, we can choose $K \in \mathbf{N}^+$ so that $z_{n_k} \in I_\ell$ for all $k \geqslant K$. But this contradicts our construction of (z_n) when $n_k \geqslant \ell$. So we conclude that $K \subseteq \bigcup_{i=1}^{k} I_i$ for some $k \in \mathbf{N}^+$. □

Hence, the following semi-constructive result gives a situation in which an anti-Specker property may be used in the place of sequential compactness. We write $B(x, r)$ for the open ball $(x - r, x + r)$.

Proposition 3.12: BISH $+$ $\mathbf{AS}^{\mathbf{ltd}}$ $\vdash$ $\mathbf{HB}^0_K$ for each inhabited, compact $K \subset \mathbf{R}$.

Proof. Fix an inhabited, compact subset $K \subset \mathbf{R}$, choose $\xi \in \mathbf{R}$ such that $\rho(\xi, K) > 0$, and suppose $(I_n)_{n\geqslant 1}$ is a sequence of inhabited, bounded open intervals (a_n, b_n) such that $K \subseteq \bigcup_{i=1}^{\infty} I_i$. For each n and appropriate $\epsilon > 0$, denote by $I_n^-(\epsilon)$ the *deflated* interval $(a_n + \epsilon, b_n - \epsilon)$. Now for each n, let

$$\epsilon_n \equiv \min\left(\left\{2^{-n}\right\} \cup \left\{\tfrac{1}{8}|I_i| :\ i \leqslant n\right\}\right),$$

and construct a finite ϵ_n-approximation $Y_n \equiv \{y_1, y_2, \ldots, y_\nu\}$ to K. By cotransitivity, each point in Y_n either belongs to $\bigcup_{i=1}^{n} I_i^-(\epsilon_n)$, or does not belong to $\bigcup_{i=1}^{n} I_i^-(2\epsilon_n)$. Hence we can define a sequence $(z_n)_{n\geqslant 1}$ in $K \cup \{\xi\}$ so that

- ❖ if $z_n \in K$, then there exists $y \in Y_n$ such that $y \notin \bigcup_{i=1}^{n} I_i^-(2\epsilon_n)$ and $z_n = y$; and
- ❖ if $z_n = \xi$, then $y \in \bigcup_{i=1}^{n} I_i^-(\epsilon_n)$ for all $y \in Y_n$.

It will turn out that if $z_k = \xi$, then $\bigcup_{i=1}^{k} I_i$ is a cover for K; hence we aim to apply $\mathbf{AS}^{\mathbf{ltd}}$ to (z_n) to obtain our desired result. One may be tempted to simplify matters by setting up (z_n) to instead

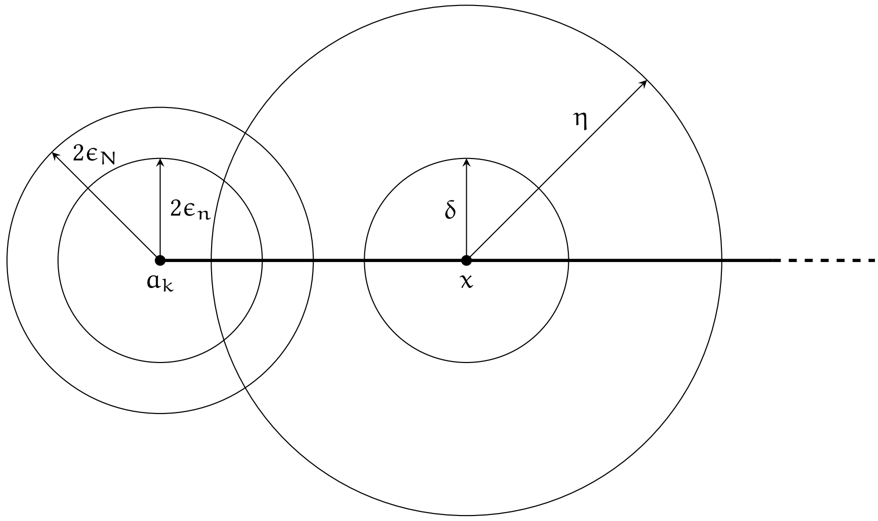


Figure 3.1: The relationship between the distances η, δ, ϵ_N and ϵ_n, within the interval I_k

decide between $y \notin \bigcup_{i=1}^{n} I_i^-(\epsilon_n)$ and $y \in \bigcup_{i=1}^{n} I_i$; however, with this approach, $z_k = \xi$ merely yields $Y_k \subseteq \bigcup_{i=1}^{k} I_i$ rather than the required $K \subseteq \bigcup_{i=1}^{k} I_i$.

Fix any $x \in K$: we show that (z_n) is eventually bounded away from x. Since $K \subseteq \bigcup_{i=1}^{\infty} I_i$, there exists $k \in \mathbf{N}^+$ such that $x \in I_k \equiv (a_k, b_k)$; furthermore, since I_k is an open interval, we can find $\eta > 0$ such that $B(x, \eta) \subseteq I_k$. Choose a number $N \geqslant k$ with $2^{-N+1} < \eta$, and let $\delta = \min\{\eta - 2\epsilon_N, \rho(\xi, K)\}$ (note that, since $\epsilon_N \leqslant 2^{-N}$, we have $\delta > 0$).

Now consider any z_n with $n \geqslant N$: we will show that $|z_n - x| \geqslant \delta$. If $z_n = \xi$, then we are done, so suppose $z_n \in K$. By definition then, we have $z_n \notin \bigcup_{i=1}^{n} I_i^-(2\epsilon_n)$, and since $n \geqslant N \geqslant k$, it follows that $z_n \notin I_k^-(2\epsilon_n)$ and thus $z_n \notin I_k^-(2\epsilon_N)$. But $B(x, \delta) \subseteq I_k^-(2\epsilon_N)$: for,

$$x - \delta \geqslant x - (\eta - 2\epsilon_N) = (x - \eta) + 2\epsilon_N \geqslant a_k + 2\epsilon_N,$$

and similarly, $x + \delta \leqslant b_k - 2\epsilon_N$. (The relationship between the distances η, δ, ϵ_N and ϵ_n is depicted in Figure 3.1.) So $|z_n - x| \geqslant \delta$, and we have shown that (z_n) is eventually bounded away from x.

Hence by $\mathbf{AS}^{\mathbf{ltd}}$, we have $z_k = \xi$ for some $k \in \mathbf{N}^+$. We now show that $K \subseteq \bigcup_{i=1}^{k} I_i$: take any $x \in K$. Then by the definition of (z_n), there exists $y \in Y_k$ such that $|x - y| < \epsilon_k$ and $y \in I_\ell^-(\epsilon_k)$ for some $\ell \leqslant k$. It follows that $x \in I_\ell$. □

A theorem of Berger and Bridges [BB08b, pp. 585–586] states that the full anti-Specker property implies a Heine-Borel property weaker than $\mathbf{HB}^0$ (namely, the *countable Heine-Borel property for disjoint intervals*, which we shall encounter in Section 3.5), and in fact requires only $\mathbf{AS}^{\mathbf{ltd}}$. **Proposition 3.12** thus improves upon this result by strengthening its conclusion. Note also that our argument here adapts to the more general situation where K is a compact subset of some arbitrary metric space, and (I_n) is a sequence of open balls within that space.

Diener has shown in [Die08, pp. 41–43] that $\mathbf{HB}^0_{[0,1]}$ and $\mathbf{FT}_\Delta$ are equivalent over **BISH**. So we have again proved **Corollary 3.10** (though now from a rather different angle).

3.3 The Non-Specker Property

The second weak anti-Specker property we will consider is the *non-Specker property* $\mathbf{AS}^{\neg}_X$. This is just the denial of **Speck** and states, for a metric space X:

> $\mathbf{AS}^{\neg}_X$ If $(z_n)_{n\geqslant 1}$ is a sequence in X, then it is impossible for (z_n) to be eventually bounded away from each point of X.

It is straightforward to show that this is equivalent to the following property, where $X \cup \{\xi\}$ is a one-point extension of X (as in **AS** and $\mathbf{AS}^{\mathbf{ltd}}$):

> If $(z_n)_{n\geqslant 1}$ is a sequence in $X \cup \{\xi\}$ that is eventually bounded away from each point of X, then
> $$(\forall i)\neg\neg(\exists n > i)\left[z_n = \xi\right].$$

Where the limited and full anti-Specker properties depended upon the choice of the parent space Z, formulations of the non-Specker property are fully determined by the choice of X. Accordingly, not just most but *all* of its significant variations lie within a single equivalence class, $\mathbf{AS}^{\neg}$.

Proposition 3.13: **BISH** $\vdash$ The following are equivalent.

(i) $\mathbf{AS}^{-}_{[0,1]}$.

(ii) $\mathbf{AS}^{-}_{\mathbf{2}^{\mathbf{N}^+}}$.

(iii) $\mathbf{AS}^{-}_{X}$ for each inhabited compact space X.

Proof. As in **Proposition 3.1**, (iii) $\implies$ (i) is trivial and (i) $\implies$ (ii) follows from **Proposition 3.2**. For (ii) $\implies$ (iii), again let X be a compact metric space, and use **Theorem 1.4** of [BR87, p. 106] to find a continuous mapping of $\mathbf{2}^{\mathbf{N}^+}$ onto X. **Lemma 3** of [BD10, p. 436] now gives the desired result. $\square$

Notice that we may also prove the implication (i) $\implies$ (ii) using the (somewhat less elegant) original construction from page 25. For, suppose there exists a Specker sequence $(\alpha_n)_{n\geqslant 1}$ in $\mathbf{2}^{\mathbf{N}^+}$. As before, define the sequence $(x_n)_{n\geqslant 1}$ in $[0,1]\cup\{2\}$ by

$$x_n = \begin{cases} F(u_n) & \text{if } u_n = \overline{\alpha_{|u_n|}}(|u_n|), \text{ and} \\ 2 & \text{otherwise.} \end{cases}$$

Now consider the sequence $(y_n)_{n\geqslant 1}$ defined by

$$y_n = \begin{cases} x_n & \text{if } x_n \in [0,1], \text{ and} \\ y_{n-1} & \text{otherwise,} \end{cases}$$

where $y_0 = 0$. It is not hard to show that (y_n) is a Specker sequence in $[0,1]$: fix any point $x \in [0,1]$. Then there exists $N \in \mathbf{N}^+$ and $\delta > 0$ such that $|x_n - x| > \delta$ for all $n \geqslant N$ [BB07, p. 200]. Consider all the paths u_m with $|u_m| = |u_N| + 1$ (where $(u_n)_{n\geqslant 1}$ is again the one-one enumeration $(\lambda, 0, 1, 00, 01, 10, 11, 000, \ldots)$ of $\mathbf{2}^*$): we must have $u_M = \overline{\alpha_{|u_M|}}(|u_M|)$ for one of these. Then $y_M = x_M$ and thus, for all $m \geqslant M$, there exists some $n \geqslant M$ such that $y_m = x_n$. But $M > N$, so it follows that $|y_m - x| > \delta$ for all $m \geqslant M$.

So we see again that the existence of a Specker sequence in $\mathbf{2}^{\mathbf{N}^+}$ entails the existence of a Specker sequence in $[0,1]$; contraposing, we obtain the desired result.

In Chapter 4, we will study a number of principles that are equivalent to the Specker property $\mathbf{Speck}_{[0,1]}$. This will allow us to identify, by contraposition, several *negative* principles that are equivalent to $\mathbf{AS}^{\neg}$: most notably, we will see in **Corollary 4.5** that $\mathbf{AS}^{\neg}$ falls into an equivalence class with several *weak fan theorems*. It would be illuminating to identify a *positive* result equivalent to $\mathbf{AS}^{\neg}$; however, it is not clear how such an equivalence could be found.

The non-Specker property is also significant in that it is the strongest anti-Specker property that we have been able to show follows *directly* from **WLPO** [BDMJ12]. While we know already that all of the anti-Specker properties we will be interested in follow from **WLPO**[5] (and, indeed, **LLPO**, due to a result of Diener [Die13]), direct proofs enable us to better understand precisely *how* these principles interact with each other — and in this case, some concrete payoff comes as soon as **Proposition 3.22**.

We begin this proof by extending **WLPO** to predicates on $\mathbf{N}^{+} \times \mathbf{N}^{+}$ (as opposed to binary sequences, which correspond to predicates on $\mathbf{N}^{+}$).

Lemma 3.14: $\mathbf{BISH} + \mathbf{WLPO} \vdash$ If P is a decidable predicate on $\mathbf{N}^{+} \times \mathbf{N}^{+}$, then

$$(\forall i)\neg\neg(\exists m)P(i,m) \vee \neg\neg(\exists i)(\forall m)\neg P(i,m).$$

Proof. For each $i \in \mathbf{N}^{+}$, define a binary sequence $(\lambda_m^{(i)})_{m\geqslant 1}$ such that

$$\begin{aligned} \lambda_m^{(i)} = 0 &\implies \neg P(i,m), \text{ and} \\ \lambda_m^{(i)} = 1 &\implies \quad P(i,m). \end{aligned}$$

WLPO allows us to decide whether or not $\boldsymbol{\lambda}^{(i)} = \mathbf{0}$; accordingly, we can define another binary sequence $(\mu_i)_{i\geqslant 1}$ so that

$$\begin{aligned} \mu_i = 0 &\implies \neg(\forall m)\left[\lambda_m^{(i)} = 0\right], \text{ and} \\ \mu_i = 1 &\implies \quad (\forall m)\left[\lambda_m^{(i)} = 0\right]. \end{aligned}$$

Now apply **WLPO** to (μ_i) to obtain the following two cases:

[5] For, over **BISH**: **WLPO** $\implies$ $\mathbf{FT}_{\Pi_1^0}$ (as we will observe in **Proposition 5.3**), and $\mathbf{FT}_{\Pi_1^0} \implies \mathbf{FT}_c \implies \mathbf{AS}$ [BB07].

❖ If $\mu_i = 0$ for all $i \in \mathbf{N}^+$, then

$$(\forall i)\neg(\forall m)\neg P(i, m),$$
$$\text{and so } (\forall i)\neg\neg(\exists m)P(i, m).$$

❖ Otherwise, we have

$$\neg(\forall i)\neg(\forall m)\neg P(i, m),$$
$$\text{and so } \neg\neg(\exists i)(\forall m)\neg P(i, m). \qquad \square$$

We now turn to our main result. The notation $\#X$ refers to the cardinality of the set X.

Proposition 3.15: BISH + WLPO $\vdash$ AS$^{\neg}$.

Proof. Let $(z_n)_{n\geqslant 1}$ be a Specker sequence in $[0, 1]$. We construct, inductively, a sequence $(I_n)_{n\geqslant 0}$ of intervals such that for each n, two properties hold.

(i) $|I_n| = 2^{-n}$, and $I_n \subset I_{n-1}$ if $n \geqslant 1$;

(ii) $(\forall i)\neg\neg(\exists m)\left[\#\left\{j \leqslant m\colon\ z_j \in I_n\right\} \geqslant i\right]$.

Most of the work of this proof revolves around property (ii), which is a weakening of the claim that the number of terms of (z_n) in I_n is unbounded.

Start the induction by setting $I_0 = [0, 1]$, which clearly satisfies (i). Furthermore, $z_j \in I_0$ for all $j \in \mathbf{N}^+$, so given any $i \in \mathbf{N}^+$, choosing $m \geqslant i$ yields $\#\{j \leqslant m\colon\ z_j \in I_0\} \geqslant i$. That is,

$$(\forall i)(\exists m)\left[\#\left\{j \leqslant m\colon\ z_j \in I_0\right\} \geqslant i\right]$$

and property (ii) follows for this base case.

Now fix any $k \in \mathbf{N}$ and suppose that we have constructed an interval $I_k = [a_k, b_k]$ with the relevant properties. Denote by ξ_k the midpoint of I_k, and let H_L and H_R be the left and right closed halves of I_k, respectively. We will make our choice of I_{k+1} from these halves: clearly, no matter which half we choose, property (i) will hold.

Since (z_n) is a Specker sequence, it is eventually bounded away from ξ_k and b_k: that is, there exist $N \in \mathbf{N}^+$ and $\delta > 0$ such that $|z_n - \xi_k| > \delta$ and $|z_n - b_k| > \delta$ for all $n \geqslant N$. Hence for any $n \geqslant N$, we can decide whether or not $z_n \in H_R$. This means that the predicate

$$P(i, m) \equiv \left(\# \left\{j : \ N \leqslant j \leqslant N + m \wedge z_j \in H_R\right\} \geqslant i\right)$$

on $\mathbf{N}^+ \times \mathbf{N}^+$ is decidable, so by **Lemma 3.14**, we can distinguish between the following two cases:

- In the case

$$(\forall i)\neg\neg(\exists m)\left[\# \left\{j : \ N \leqslant j \leqslant N + m \wedge z_j \in H_R\right\} \geqslant i\right], \tag{3.1}$$

  set $I_{k+1} = H_R$. Fix any $i \in \mathbf{N}^+$ and assume that

$$\neg(\exists m)\left[\# \left\{j \leqslant m : \ z_j \in I_{k+1}\right\} \geqslant i\right], \tag{3.2}$$
$$\text{and so } (\forall m)\neg\left[\# \left\{j \leqslant N + m : \ z_j \in I_{k+1}\right\} \geqslant i\right].$$

  Then since

$$\# \left\{j : \ N \leqslant j \leqslant N + m \wedge z_j \in I_{k+1}\right\} \geqslant i$$
$$\implies \# \left\{j \leqslant N + m : \ z_j \in I_{k+1}\right\} \geqslant i$$

  for all m, we have

$$(\forall m)\neg\left[\# \left\{j : \ N \leqslant j \leqslant N + m \wedge z_j \in I_{k+1}\right\} \geqslant i\right].$$

  But this contradicts (3.1); hence we obtain $\neg$(3.2), and since i was arbitrary, property (ii) follows for I_{k+1}.

- In the case

$$\neg\neg(\exists i)(\forall m)\left[\# \left\{j : \ N \leqslant j \leqslant N + m \wedge z_j \in H_R\right\} < i\right], \tag{3.3}$$

it is impossible for there to be infinitely many m with $z_m \in H_R$. Set $I_{k+1} = H_L$, and assume both

$$(\exists i)(\forall m)\left[\#\{j\colon\ N \leqslant j \leqslant N+m \wedge z_j \in H_R\} < i\right] \tag{3.4}$$

$$\text{and } (\exists i)\neg(\exists m)\left[\#\left\{j \leqslant m\colon\ z_j \in I_{k+1}\right\} \geqslant i\right]. \tag{3.5}$$

Accordingly, construct numbers $i_1, i_2 \in \mathbf{N}^+$ such that

$$(\forall m)\left[\#\{j\colon\ N \leqslant j \leqslant N+m \wedge z_j \in H_R\} < i_1\right] \tag{3.6}$$

$$\text{and } (\forall m)\left[\#\left\{j \leqslant m\colon\ z_j \in I_{k+1}\right\} < i_2\right]. \tag{3.7}$$

We can weaken (3.7) to obtain

$$(\forall m)\left[\#\left\{j\colon\ N \leqslant j \leqslant N+m \wedge z_j \in I_{k+1}\right\} < i_2\right]. \tag{3.8}$$

Now from our choice of N we see that, for all $n \geqslant N$, we have $z_n \in I_k$ only if either $z_n \in H_L \equiv I_{k+1}$ or $z_n \in H_R$. Hence we can combine (3.6) and (3.8) to obtain:

$$(\forall m)\left[\#\left\{j\colon\ N \leqslant j \leqslant N+m \wedge z_j \in I_k\right\} < i_1 + i_2\right]. \tag{3.9}$$

But it follows from induction assumption (ii) that

$$\neg(\forall m)\left[\#\left\{j \leqslant m\colon\ z_j \in I_k\right\} < i_1 + i_2 + (N-1)\right],$$

and since

$$\begin{aligned}
&\#\left\{j\colon\ N \leqslant j \leqslant N+m \wedge z_j \in I_k\right\} < i_1 + i_2 \\
&\qquad \Longrightarrow\ \#\left\{j \leqslant N+m\colon\ z_j \in I_k\right\} < i_1 + i_2 + (N-1) \\
&\qquad \Longrightarrow\ \#\left\{j \leqslant m\colon\ z_j \in I_k\right\} < i_1 + i_2 + (N-1)
\end{aligned}$$

for all m, we have

$$\neg(\forall m)\left[\#\left\{j\colon\ N \leqslant j \leqslant N+m \wedge z_j \in I_k\right\} < i_1 + i_2\right],$$

which contradicts (3.9). So we have (3.5) $\implies \neg$(3.4). But (3.3) $\equiv \neg\neg$(3.4); hence we conclude $\neg$(3.5), and property (ii) follows for I_{k+1}.

This completes the construction of (I_n). Now, it follows from property (i) that $\bigcap_{n\geqslant 1} I_n$ consists of a single point — call it ξ. Pick $N \in \mathbf{N}^+$ and $\delta > 0$ such that $|z_n - \xi| > \delta$ for all $n \geqslant N$, and choose k for which $|I_k| < \frac{1}{2}\delta$. Now for all $n \geqslant N$, we have

$$\left|z_n - \xi_k\right| \geqslant \left|z_n - \xi\right| - \left|\xi - \xi_k\right| > \delta - \tfrac{1}{2}\delta = \tfrac{1}{2}\delta$$

(where, as earlier, ξ_k denotes the midpoint of the interval I_k). Therefore $z_n \notin I_k$. It then follows that

$$\neg(\exists m)\left[\#\left\{j \leqslant m\colon\ z_j \in I_k\right\} \geqslant N+1\right],$$

which contradicts property (ii) of I_k in the case $i \equiv N+1$. □

3.4 Non-Oscillation Properties

The requirement that a sequence be eventually bounded away from *every* point in a space rules out both oscillation[6] and convergence. There may be something interesting to observe if we instead adopt the following, weaker requirement, which still rules out oscillation but allows convergence: say that a sequence (x_n) in a metric space X *has at most one limit* (in X) if, for every pair a, b of distinct points in X, either (x_n) is eventually bounded away from a, or (x_n) is eventually bounded away from b.

It seems as though this property of having at most one limit could be used in place of convergence to weaken strongly nonconstructive principles, yielding versions of greater constructive interest. Consider, for example, the *Bolzano-Weierstraß principle*, which instantiates sequential compactness:

BWP Every bounded sequence of real numbers has a convergent subsequence.

Replacing convergence with the property of having at most one limit, we obtain the following, seemingly weaker, principle:

[6] A sequence may be considered to *oscillate* if it has at least two cluster points.

WBWP Every bounded sequence of real numbers has a subsequence with at most one limit.

However, not all is as it seems here.

Proposition 3.16: BISH + WBWP $\vdash$ LPO.

Proof. Fix any binary sequence $(\lambda_n)_{n\geqslant 1}$, and define a (bounded) sequence $(z_n)_{n\geqslant 1}$ in $[0,1]$ so that

$$z_n = 0 \implies \lambda_m = 0 \text{ for all } m \leqslant n, \text{ and}$$
$$z_n = 1 \implies \lambda_m = 1 \text{ for some } m \leqslant m.$$

Use **WBWP** to construct a subsequence $(z_{n_k})_{k\geqslant 1}$ of (z_n) with at most one limit. Two cases arise:

- If (z_{n_k}) is eventually bounded away from 0, then $z_{n_k} = 1$ eventually, whence $\lambda_m = 1$ for some m.
- If (z_{n_k}) is eventually bounded away from 1, then $z_{n_k} = 0$ eventually. Then for each $m \in \mathbf{N}^+$, we may find $n_k \geqslant m$ with $z_{n_k} = 0$, whence $\lambda_m = 0$. □

Corollary 3.17: BISH $\vdash$ The following are equivalent.

(i) **WBWP**.

(ii) **BWP**.

(iii) **LPO**.

Proof. Mandelkern has shown [Man88] that **LPO** $\implies$ **BWP**, and clearly **BWP** $\implies$ **WBWP**. Proposition 3.16 then collapses these three principles. □

So this "weakening" of **BWP** is not a weakening at all!

A more interesting use of this property of having at most one limit is to give grounds on which, having ruled out oscillation, one can decide whether a sequence converges or is a Specker sequence. We thus formulate the *limit-stability property* for the unit interval:

$\mathbf{LSP}_0$ If (x_n) is a sequence in $[0, 1]$ that has at most one limit, then (x_n) either converges in $[0, 1]$ or is eventually bounded away from each point of $[0, 1]$.

It seems *prima facie* that $\mathbf{LSP}_0$ ought to be weaker than **AS**. Both of these principles concern the classification of sequences into three classes: convergent, Specker, and oscillatory. But where the anti-Specker properties explicitly rule out (often in some strong sense!) one of these possibilities, the limit-stability property merely allows one to decide between these three types of behaviour (in a sense weaker than a direct disjunction). Again, though, things are not as they seem.

Proposition 3.18: $\mathbf{BISH} + \mathbf{LSP}_0 \vdash \mathbf{AS}$.

Proof. Let $(z_n)_{n \geqslant 1}$ be a sequence in $[0, 1] \cup \{2\}$ that is eventually bounded away from each point of $[0, 1]$. Define another sequence $(x_n)_{n \geqslant 1}$ as follows: for each $n \in \mathbf{N}^+$, set $x_{2n-1} = \frac{1}{2} z_n$ and $x_{2n} = 1$. We show that (x_n) has at most one limit in $[0, 1]$. Fix any two distinct points $a, b \in [0, 1]$. Then either $a \neq 1$ or $b \neq 1$: suppose, without loss of generality, that $a \neq 1$, and pick $\epsilon_1 \in (0, 1 - a)$.

Since (z_n) is eventually bounded away from 1, there exist $\epsilon_2 > 0$ and $N \in \mathbf{N}^+$ such that $\left|x_n - \frac{1}{2}\right| > \epsilon_2$ for all $n \geqslant N$. Two cases now arise:

- If $a \in \left[0, \frac{1}{2}\right]$, then $\left(\frac{1}{2} z_n\right)_{n \geqslant 1}$ is eventually bounded away from a, and therefore so is (x_n).
- If $a \in \left[\frac{1}{2} - \frac{1}{2}\epsilon_2, 1 - \frac{1}{2}\epsilon_1\right]$, then $|x_n - a| \geqslant \frac{1}{2} \min\{\epsilon_1, \epsilon_2\}$ for all $n \geqslant N$.

In either case, (x_n) is eventually bounded away from a; hence it has at most one limit. It then follows from $\mathbf{LSP}_0$ that (x_n) is *limit-stable:* that is, it either converges to some limit in $[0, 1]$ or is eventually bounded away from each point of $[0, 1]$. The latter case cannot hold, since (x_n) is clearly not eventually bounded away from 1; hence there exists $x \in [0, 1]$ with $x_n \to x$.

But $x_{2n} \to 1$, so we must have $x = 1$. We thus have the subsequence $\left(\frac{1}{2} z_n\right)_{n \geqslant 1}$ converging to 1, whence $z_n = 2$ eventually. □

A surprising consequence of this result is that the original property $\mathbf{LSP}_0$ is equivalent to the following, seemingly stronger, formulation, which we shall henceforth use in its stead:

LSP If (x_n) is a sequence in $[0, 1]$ that has at most one limit, then (x_n) converges in $[0, 1]$.

This highlights the similarity of the limit-stability property to two others in the CRM literature; namely, the constructive definition of sequential compactness used in [BIS99], and the principle **SC** of [BB08b, p. 587].

Now that we have some idea of the sort of principles entailed by **LSP**, it would be desirable to identify stronger principles from which it follows. Of some interest here is the following *strong limit-stability property*, which requires only that (x_n) has at most one limit in a weaker (essentially double-negated) sense.

> **SLSP** If (x_n) is a sequence in $[0, 1]$ such that for all distinct points $a, b \in X$, it is impossible for (x_n) to be eventually bounded away from neither a nor b, then (x_n) converges in $[0, 1]$.

It turns out that, as we now show, **SLSP** is equivalent to **LPO**.

Proposition 3.19: BISH + LPO $\vdash$ SLSP.

Proof. Fix any sequence $(x_n)_{n\geqslant 1}$ in $[0, 1]$ with the property that for all distinct $a, b \in [0, 1]$, it is impossible for (x_n) to be eventually bounded away from neither a nor b. We can use **BWP** to find a convergent subsequence $(x_{n_k})_{k\geqslant 1}$: let x be the limit of this subsequence. Suppose that

$$(\exists \epsilon > 0)(\forall N)(\exists n \geqslant N)\big[|x_n - x| \geqslant \epsilon\big]. \tag{3.10}$$

Fixing such an ϵ, we obtain a subsequence $(x_{m_k})_{k\geqslant 1}$ that is bounded away from x (by ϵ). This subsequence must itself have a convergent subsequence $(x_{m_{k_\ell}})_{\ell\geqslant 1}$: let x' be its limit. But then $x \neq x'$, and (x_n) is eventually bounded away from neither x nor x' — a contradiction. So we have $\neg$(3.10), and with several more applications of **LPO** conclude

$$(\forall \epsilon > 0)(\exists N)(\forall n \geqslant N)\big[|x_n - x| < \epsilon\big].$$

That is, (x_n) converges to $x \in [0, 1]$, and we have shown that **SLSP** holds. □

Proposition 3.20: BISH + SLSP $\vdash$ LPO.

Proof. Fix any sequence $(x_n)_{n\geqslant 1}$ in $[0,1]$, and consider the following weak convergence properties, originally formulated by Heyting [Hey71, p. 113]:

$$(\forall k)\neg\neg(\exists N)(\forall n > N)\left[|x_n - x_N| < 2^{-k}\right]; \tag{3.11}$$

$$(\exists x)(\forall k)\neg\neg(\exists N)(\forall n > N)\left[|x_n - x| < 2^{-k}\right]. \tag{3.12}$$

(i) Suppose that (3.11) holds, and fix any two distinct points $a, b \in [0,1]$ with $a < b$. Choosing k so that $2^{-k} \leqslant \frac{1}{4}(b-a)$, we have

$$\neg\neg(\exists N)(\forall n > N)\left[|x_n - x_N| < \tfrac{1}{4}(b-a)\right]. \tag{3.13}$$

Now assume that

$$(\exists N)(\forall n > N)\left[|x_n - x_N| < \tfrac{1}{4}(b-a)\right].$$

Fix N and consider x_N: either $x_N > a + \frac{1}{3}(b-a)$ or $x_N < a + \frac{2}{3}(b-a)$. In the former case, we have, for all $n > N$:

$$\begin{aligned} x_n - a &= (x_n - x_N) + (x_N - a) \\ &> -\tfrac{1}{4}(b-a) + \tfrac{1}{3}(b-a) \\ &= \tfrac{1}{12}(b-a) > 0. \end{aligned}$$

Hence (x_n) is eventually bounded away from a. A similar argument shows that (x_n) is eventually bounded away from b in the latter case. So, we have shown that

$$(\exists N)(\forall n > N)\ \left[|x_n - x_N| < \tfrac{1}{4}(b-a)\right] \implies \left(\mathbf{EBA}(a) \vee \mathbf{EBA}(b)\right),$$

where $\mathbf{EBA}(w)$ stands for "(x_n) is eventually bounded away from w." Contraposing twice and applying (3.13), we obtain $\neg\neg\left(\mathbf{EBA}(a)\vee\mathbf{EBA}(b)\right)$: this holds for all distinct $a, b \in [0,1]$. Hence (x_n) has at most one limit in the weak sense of **SLSP**.

(ii) If (x_n) converges to some limit $x \in [0,1]$, then we have

$$(\exists x)(\forall k)(\exists N)(\forall n > N)\left[|x_n - x| < 2^{-k}\right],$$

which clearly entails the weaker claim (3.12).

Combining (i) and (ii), we see that

$$\mathbf{BISH} + \mathbf{SLSP} \ \vdash\ (3.11) \implies (3.12).$$

But a proof of Heyting [Hey71, p. 114] shows that this implication entails **LPO**. □

Corollary 3.21: BISH $\vdash$ The following are equivalent.

(i) **SLSP**.

(ii) **LPO**.

Proof. Follows from **Proposition 3.20** and **Proposition 3.19**. □

As **SLSP** appears to be a proper strengthening of **LSP**, its equivalence with **LPO** suggests that **LSP** may be strictly weaker than **LPO**. Indeed, we can prove that this is the case by reprising the construction used to prove **Proposition 3.15**. This is something of a demonstration of the utility of direct proofs: while the statement itself of **Proposition 3.15** was known to be true, the novel proof we gave now facilitates a new result.

Proposition 3.22: BISH + WLPO $\vdash$ **LSP**.

Proof. Fix any sequence $(z_n)_{n\geqslant 1}$ in $[0,1]$ that has at most one limit. As in **Proposition 3.15**, we inductively construct a diminishing sequence of nested intervals. However, now that we have only a sequence with at most one limit — rather than a sequence eventually bounded away from each point of $[0,1]$ — we cannot necessarily split these intervals about their midpoints. Instead, given the interval $I_k = [a_k, b_k]$, consider $\xi_k^L = a_k + \frac{1}{3}|I_k|$ and $\xi_k^R = a_k + \frac{2}{3}|I_k|$. Through repeated use of (z_n)'s having at most one limit, we see that (z_n) is eventually bounded away from (at least) three of a_k, ξ_k^L, ξ_k^R and b_k; accordingly, we split up I_k as follows:

- ❖ If (z_n) is eventually bounded away from b_k and some $\xi_k \in \{\xi_k^L, \xi_k^R\}$, take $H_R \equiv [\xi_k, b_k]$ and $H_L \equiv [a_k, \xi_k]$.
- ❖ Otherwise, (z_n) is eventually bounded away from a_k, ξ_k^L and ξ_k^R, so take $H_R \equiv [a_k, \xi_k^L]$ and $H_L \equiv [\xi_k^L, b_k]$.

(That is: H_R no longer necessarily lies to the right of H_L, but still has the crucial property that we can eventually determine whether terms of (z_n) lie within it or not, so that the relevant predicate P is decidable.)

The argument of **Proposition 3.15** now yields a sequence of intervals $(I_n)_{n \geqslant 0}$ with properties that, in turn, allow us to find a point $\xi \in \bigcap_{n \geqslant 1} I_n$ such that (z_n) cannot be eventually bounded away from ξ. Hence, (z_n) is eventually bounded away from every $w \neq \xi$ in $[0, 1]$. We now show that (z_n) converges to ξ.

Fix any $\epsilon \in \left(0, \frac{1}{2}\right)$. We aim to show that (z_n) is eventually contained within the closed ball $\overline{B}(\xi, \epsilon) \equiv [\xi - \epsilon, \xi + \epsilon]$ by separating $[0, 1]$ into an appropriately small interval around ξ and some complementary space around this interval, then applying the anti-Specker property. However, the situation changes somewhat when ξ is near 0 or 1. We have three possibilities:

- If $0 < \xi < 1$, assume without loss of generality that ϵ is sufficiently small to satisfy $0 < \xi - \epsilon$ and $\xi + \epsilon < 1$. Find $N \in \mathbf{N}^+$ and $\delta > 0$ such that $|z_n - (\xi - \epsilon)| > \delta$ and $|z_n - (\xi + \epsilon)| > \delta$ for all $n \geqslant N$, and $0 < \xi - \epsilon - \delta$ and $\xi + \epsilon + \delta < 1$. Take

$$X \equiv \left[0, \xi - \epsilon - \delta\right] \cup \left[\xi + \epsilon + \delta, 1\right].$$

- If $\xi - \epsilon < 0$, then $\xi + \epsilon < 1$. So find $N \in \mathbf{N}^+$ and $\delta > 0$ such that $|z_n - (\xi + \epsilon)| > \delta$ for all $n \geqslant N$, and $\xi + \epsilon + \delta < 1$. Take

$$X \equiv \left[\xi + \epsilon + \delta, 1\right].$$

- If $\xi + \epsilon > 1$, make an analogous construction to that of the foregoing case.

In all cases, X is a compact set in the space $Z \equiv X \cup \overline{B}(\xi, \epsilon)$. It follows from **Proposition 1** of [Bri09b, p. 439] that the following variation upon the anti-Specker property is equivalent to our usual formulation **AS** — note the crucial requirement of X-*detachability*:

(∗) Every X-detachable sequence in Z that is eventually bounded away from each point of X is eventually not in X.

Since **AS** follows from **WLPO**, we can apply (∗) to the sequence $(z_n)_{n \geqslant N}$ in Z — which is X-detachable and eventually bounded away from each point of X — to get (z_n) in $\overline{B}(\xi, \epsilon)$ eventually. Since we can make this construction for arbitrarily small $\epsilon > 0$, we see that $z_n \to \xi$. □

While we have with this proof established that **LSP** lies somewhere between **WLPO** and **AS**, it would be worthwhile to determine its strength more precisely in future work: this would allow us to better identify the degree and nature of non-algorithmic content that it brings to proofs. It would be particularly desirable to show that it follows from one of the stronger fan theorems, thereby placing it firmly within that hierarchy of intuitionistic principles; however, such a result seems unlikely.

3.5 Increasing Anti-Specker Properties

Recall that Specker's theorem, in its original formulation, asserted the existence of a *nondecreasing* Specker sequence. To reflect this, we now acknowledge a further two weak anti-Specker properties which are nondecreasing counterparts of **AS** and $\mathbf{AS}^{-}$. The *increasing anti-Specker property* for a (compact) metric space X with total order $\leqslant$ states:

> $\mathbf{AS}_X^{\uparrow}$ If $X \cup \{\xi\}$ is a one-point extension of X with $x < \xi$ for each $x \in X$, and $(z_n)_{n \geqslant 1}$ is a *nondecreasing* sequence in $X \cup \{\xi\}$ that is eventually bounded away from each point of X, then $z_n = \xi$ eventually.

We will consider the following more general formulation, which follows readily from $\mathbf{AS}^{\mathbf{ltd}}$:

> $\mathbf{AS}^{\uparrow}$ $\mathbf{AS}_X^{\uparrow}$ holds for every totally ordered compact metric space X.

It may be that $\mathbf{AS}^{\uparrow}$ and (say) $\mathbf{AS}_{[0,1]}^{\uparrow}$ are equivalent, as was the case for the anti-Specker properties examined earlier in this chapter. However, determining the fact of this matter remains an open problem: the need to preserve the ordering of the relevant sequences complicates the situation somewhat.

By the same token, we have the following *increasing non-Specker properties*:

$\mathbf{AS}_X^{\uparrow\neg}$ If $(z_n)_{n\geqslant 1}$ is a *nondecreasing* sequence in X, then it is impossible for (z_n) to be eventually bounded away from each point of X.

$\mathbf{AS}^{\uparrow\neg}$ $\mathbf{AS}_X^{\uparrow\neg}$ holds for every totally ordered compact metric space X.

These properties are very natural to formulate and, indeed, constitute more faithful antitheses of Specker's theorem than our prior anti-Specker properties; however, they are much harder to apply than, say, $\mathbf{AS}^{\mathbf{ltd}}$. Nevertheless, we are able to use $\mathbf{AS}^{\uparrow}$ in a compactness-like role to establish another Heine-Borel property; namely, the (very weak) *countable Heine-Borel property for disjoint intervals* of [BB08b, p. 585]. This states, for a subset $K \subset \mathbf{R}$:

$\mathbf{HB}_K^{\mathrm{di}}$ If $(I_n)_{n\geqslant 1}$ is a sequence of pairwise-disjoint, inhabited, bounded open intervals such that $K \subseteq \bigcup_{i=1}^{\infty} I_i$, then there exists $k \in \mathbf{N}^+$ such that $K \subseteq \bigcup_{i=1}^{k} I_i$.

We will prove that this property holds for compact subsets $K \subset \mathbf{R}$; to do so, we make use of the following lemma.

Lemma 3.23: BISH $\vdash$ Let $(I_n)_{n\geqslant 1}$ be a sequence of pairwise-disjoint, inhabited, bounded open intervals, and K a compact subset of $\bigcup_{i=1}^{\infty} I_i$. Then for each $\ell \in \mathbf{N}^+$, either $K \setminus I_\ell = \varnothing$, or $K \setminus I_\ell$ is inhabited and compact.

Proof. Fix ℓ and write $I_\ell \equiv (a, b)$. By Bishop's lemma (**Proposition 3.1.1** of [BV06, p. 64]), there exists $x \in K$ such that if $a \neq x$, then $\rho(a, K) > 0$. Find k such that $x \in I_k$. Were it the case that $a \in I_k$, we would be able to find a point (slightly greater than a) belonging to both I_ℓ and I_k, contradicting the hypothesis that these intervals are disjoint; hence, $a \notin I_k$. This separates a and x, so we see that $\rho(a, K) > 0$; a similar argument shows that $\rho(b, K) > 0$ also. Fix any positive $r < \min\{\rho(a, K), \rho(b, K)\}$.

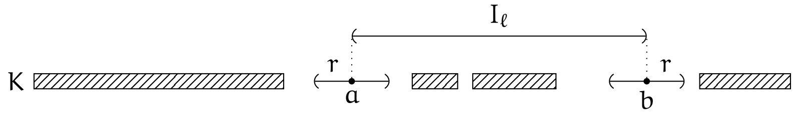


Figure 3.2: The interval I_ℓ and its endpoints, which are separated from the components of K.

Given any $x \in K$, we can decide which of the following two cases holds:

- ❖ $a - r < x < b + r$, whence $x \in I_\ell$;
- ❖ $x < a$ or $x > b$, whence $x \notin I_\ell$ (and so $x \in K \setminus I_\ell$).

(That is, I_ℓ is in a certain sense *detachable* from K.) In particular, if $\inf K \in I_\ell$ and $\sup K \in I_\ell$, we see that $K \subset I_\ell$ and thus $K \setminus I_\ell = \varnothing$. Otherwise, one of $\inf K$ and $\sup K$ belongs to $K \setminus I_\ell$, which is therefore inhabited. We henceforth assume that $K \setminus I_\ell$ is inhabited, and show furthermore that it is compact.

$K \setminus I_\ell$ **is complete.** Let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in $K \setminus I_\ell$, and let x be the limit of (x_n) in K. Were it the case that $x \in I_\ell$, we would have $|x_n - x| > 2r$ for all n — a contradiction. Hence $x \in K \setminus I_\ell$.

$K \setminus I_\ell$ **is totally bounded.** Given $\epsilon > 0$, let $\delta = \min\{\epsilon, 2r\}$, and take a finite δ-approximation Y to K. For each point $y \in Y$, determine whether $y \in I_\ell$ or $y \in K \setminus I_\ell$, and discard y in the former case. The resulting set is a finite ϵ-approximation to $K \setminus I_\ell$. □

We now prove our intended result.

Proposition 3.24: $\mathbf{BISH} + \mathbf{AS}^{\uparrow} \vdash \mathbf{HB}_K^{\mathrm{di}}$ for each inhabited, compact $K \subset \mathbf{R}$.

Proof. Fix a compact $K \subset \mathbf{R}$, choose $\xi \in \mathbf{R}$ such that $\xi - \sup K > 1$, and suppose that $(I_n)_{n\geqslant 1}$ is a sequence of intervals satisfying the hypotheses of $\mathbf{HB}_K^{\mathrm{di}}$. Construct a nondecreasing sequence $(z_n)_{n\geqslant 1} \in K \cup \{\xi\}$, along with a sequence $(K_n)_{n\geqslant 1}$ of compact subsets of K and a sequence $(\ell_n)_{n\geqslant 1}$ of indices, as follows:

Take $K_1 = K$, and let $z_1 = \inf K_1$. (Notice that, since K_1 is a totally bounded subset of **R**, **Proposition 2.2.5** of [BV06, p. 39] guarantees the existence of $\inf K_1$; then, by completeness, $\inf K_1 \in K_1$.) Now for each $n \geqslant 1$, given $K_n \subseteq K$ and $z_n = \inf K_n \in K_n$, find ℓ_n such that $z_n \in I_{\ell_n}$, and consider

$$K_{n+1} \equiv K_n \setminus I_1 \setminus I_2 \setminus \cdots \setminus I_{\ell_n}.$$

By **Lemma 3.23**, K_{n+1} is either empty, or inhabited and compact.

- If K_{n+1} is empty, set $z_m = \xi$ for all $m \geqslant n+1$. (It doesn't matter how (K_n) and (ℓ_n) behave after this point — say $K_m = \varnothing$ and $\ell_m = \ell_n$.)
- If K_{n+1} is inhabited and compact, set $z_{n+1} = \inf K_{n+1} \in K_{n+1}$ and repeat this construction. Since $K_{n+1} \subset K_n$, the sequence (z_n) thus obtained is nondecreasing.

Note also that, for each n with $z_{n+1} \in K_{n+1}$, we can show the following:

$$(*) \qquad \ell_m \geqslant m \text{ for all } m \leqslant n+1.$$

For, since $z_{n+1} \notin I_1, I_2, \ldots, I_{\ell_n}$ but $z_{n+1} \in I_{\ell_{n+1}}$, we have $\ell_{n+1} > \ell_n$. Furthermore, $z_{m+1} \in K_{m+1}$ for all m with $1 \leqslant m \leqslant n$; so, repeating the same argument a total of n times, we see that $\ell_{n+1} > \ell_n > \cdots > \ell_1 \geqslant 1$. This proves $(*)$, and also that

$$K_{n+1} = K \setminus I_1 \setminus I_2 \setminus \cdots \setminus I_{\ell_n}$$

for each $n \in \mathbf{N}^+$.

Now fix any point $x \in K$: we show that (z_n) is eventually bounded away from x. Find an index $k \in \mathbf{N}^+$ such that $x \in I_k$. Since I_k is open, there exists $r > 0$ such that $B(x, r) \subset I_k$. Now consider z_{m+1} for any $m \geqslant k$:

- If $z_{m+1} = \xi$, we have $|z_{m+1} - x| > 1$.
- If $z_{m+1} \in K_{m+1}$, then $\ell_m \geqslant m \geqslant k$ by $(*)$. In this case $z_{m+1} \notin I_1, I_2, \ldots, I_{\ell_m}$, and in particular, $z_{m+1} \notin I_k$. Hence $|z_{m+1} - x| > r$.

In either case, $|z_{m+1} - x| > \min\{1, r\}$; since this holds for all $m \geqslant k$, we conclude that (z_n) is indeed eventually bounded away from x. Applying $\mathbf{AS}^{\uparrow}$, we find some minimal index $N \in \mathbf{N}^+$ such that $z_{N+1} = \xi$. Then

$$K_{N+1} = K \setminus \bigcup_{i=1}^{\ell_N} I_i = \varnothing,$$

and we conclude that $K \subseteq \bigcup_{i=1}^{\ell_N} I_i$. □

The problem of identifying how much weaker than $\mathbf{AS}^{\mathbf{ltd}}$ and $\mathbf{AS}^{\neg}$ these increasing anti-Specker properties are remains open. That is: what principle can be added to **BISH** to allow one to pass from an increasing anti-Specker property to its less restricted analogue? This corresponds to a problem we will encounter in Chapter 4; namely, that of finding how much weaker the principle $\mathbf{Speck}_{[0,1]}$ is than its counterpart $\mathbf{Speck}^{\uparrow}_{[0,1]}$.

The results of this chapter are summarised in **Figure 4.6**; however, this figure also incorporates several *weak fan theorems* and corresponding results which we do not introduce until Section 4.1; hence, it does not appear until page 71.

4 | Recursive Considerations

In this chapter, we identify three equivalence classes of principles that are, largely, opposing counterparts of ones we have encountered already. In doing so, we will consider principles that do not hold classically or intuitionistically, but many of which are true within **RUSS**. Accordingly, the linchpins of this investigation will be two variations upon Specker's theorem for the unit interval. Recall the first of these from Section 2.1:

Speck$_{[0,1]}$ There exists a sequence in $[0, 1]$ that is eventually bounded away from each point of $[0, 1]$.

We will begin by exploring the relationship between **Speck**$_{[0,1]}$ and antitheses of various versions of Brouwer's fan theorem. We say that a bar B for $\mathbf{2}^*$ is *nonuniform* if, for each $n \in \mathbf{N}$, there exists a path of length n in $\mathbf{2}^*$ that misses B. With this in mind, we will be interested in principles of the form:

anti-FT$_?$ There exists a nonuniform ?-bar for $\mathbf{2}^*$.

Diener studied these principles in [Die08, pp. 65–70], proving (among other things) the equivalence of **anti-FT**$_c$, **anti-FT**$_{\mathrm{Full}}$ and **Speck**$_{[0,1]}$. However, the proof we present here is novel and illuminates the relationships between these principles from a new perspective. In particular, we demonstrate how the Specker property arises by appealing directly to intuitions about the behaviour of paths in the complete binary fan, rather than by using Bishop's lemma (and in doing so utilise a different embedding of paths into the unit interval[1]). The hope is that this supplements Diener's proof in giving a glimpse at the underlying structural relationships between nonuniform bars and Specker sequences.

[1] F rather than G; recall pages 25 and 26.

The disadvantage of our approach here is that it relies upon Lemma 4.1; however, this result lends itself to reasoning in a way familiar to classical mathematicians and is less steeped in constructive nuance than Bishop's lemma.

4.1 Fan-Theoretic Equivalents of $\mathbf{AS}^{\neg}$

Our exploration of the relationship between $\mathbf{Speck}_{[0,1]}$ and the various anti-fan-theorems will shed light upon a corresponding relationship between the non-Specker property $\mathbf{AS}^{\neg}$ and certain *weak fan theorems*. We start down this path by observing the following result of Berger and Bridges [BB07, p. 198] about our embedding F from Section 3.1.

Lemma 4.1: **BISH** $\vdash$ If $(z_n)_{n\geqslant 1}$ is a sequence of real numbers that is eventually bounded away from $F(\alpha)$ for each $\alpha \in \mathbf{2}^{\mathbf{N}^+}$, then (z_n) is eventually bounded away from each point of $[0, 1]$.

This allows us to prove the following result.

Proposition 4.2: **BISH** + **anti-FT**$_{\text{Full}}$ $\vdash$ $\mathbf{Speck}_{[0,1]}$.

Proof. Assuming **anti-FT**$_{\text{Full}}$, let B be a nonuniform bar: that is, for each n, there exists a path u_n of length n that misses B. Define a sequence $(z_n)_{n\geqslant 1}$ in $[0, 1]$ by $z_n = F(u_n)$. We will show that (z_n) is a Specker sequence.

Fix any infinite path $\alpha \equiv (a_n)_{n\geqslant 1}$. Since B is a bar, we can find $N_1 \in \mathbf{N}^+$ such that $\overline{\alpha}N_1 \in B$. To illustrate the next part of the proof, suppose that $a_{N_1} = 0$ (the case $a_{N_1} = 1$ is similar). Let k be the greatest integer less than N_1 such that $a_k = 1$ (if no such k exists, the desired result follows from cases (i) and (ii) of the following argument). Define infinite paths $\beta \equiv (b_n)_{n\geqslant 1}$ and $\gamma \equiv (c_n)_{n\geqslant 1}$ by:

$$b_n = \begin{cases} a_n & \text{if } n < k, \\ 0 & \text{if } n = k, \\ 1 & \text{if } n > k; \end{cases} \quad \text{and} \quad c_n = \begin{cases} a_n & \text{if } n < N_1, \\ 1 & \text{if } n = N_1, \\ 0 & \text{if } n > N_1 \end{cases}$$

(as illustrated in **Figure 4.1**). Now, find $N_2, N_3 \in \mathbf{N}^+$ such that $\overline{\beta}N_2 \in B$ and $\overline{\gamma}N_3 \in B$, and define $N \equiv \max\{N_1, N_2, N_3\}$. Fix any $n \geqslant N$: we will show that $|F(\alpha) - z_n| \geqslant 2^{-N}$. Consider the least

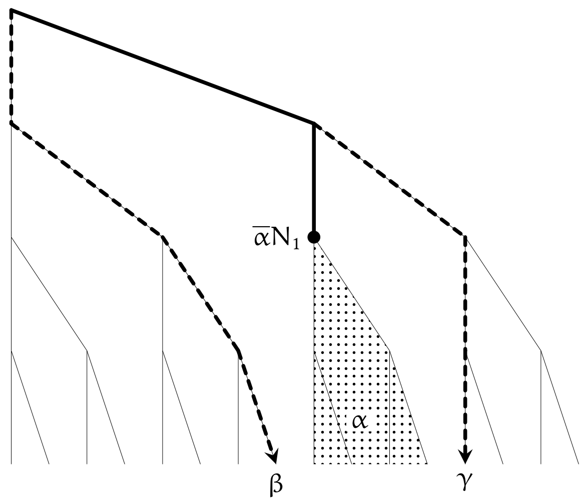
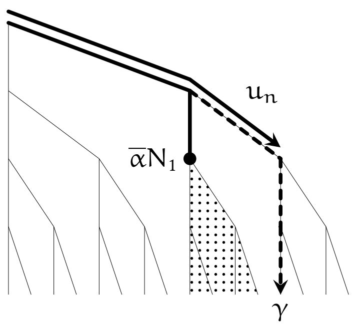


Figure 4.1: The relationship between the paths α, β and γ

integer m with $1 \leqslant m \leqslant N_1$ such that $[u_n]_m \neq a_m$ but $\overline{u_n}(m-1) = \overline{\alpha}(m-1)$ (we know that such an m exists because $|u_n| = n \geqslant N \geqslant N_1$ but $\overline{u_n}(N_1) \neq \overline{\alpha}(N_1) \in B$). Four cases arise:

(i) **Case 1:** $a_m = 0$ and $m = N_1$.



Note that

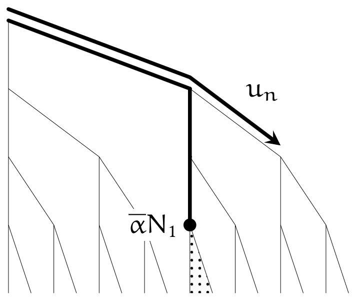
$$F(\alpha) \leqslant F(\gamma) = F(a_1, a_2, \dots, a_{N_1-1}, 1, 0, 0, \dots).$$

Since $\overline{\gamma}N_3 \in B$ and u_n is not blocked by B, we have

$$F(u_n) \geqslant F(a_1, a_2, \dots, a_{N_1-1}, 1, 0, 0, \dots, 0, \underset{\underset{N_3}{\uparrow}}{1}, 0, 0, \dots);$$

hence we deduce that $F(u_n) - F(\alpha) \geqslant 2^{-N_3} \geqslant 2^{-N}$.

(ii) **Case 2:** $\alpha_m = 0$ and $m < N_1$.



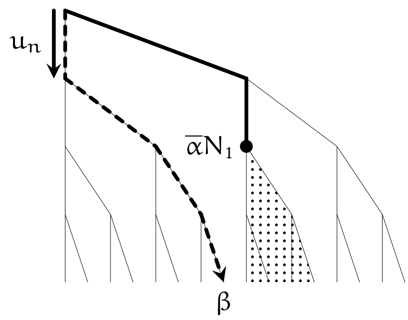
In this case we have:

$$
\begin{aligned}
F(u_n) &\geqslant F(a_1, a_2, \dots, a_{m-1}, 1, 0, 0, \dots), \\
\text{and } F(\alpha) &\leqslant F(a_1, a_2, \dots, a_{m-1}, 0, 1, 1, \dots, 1, \underset{\substack{\uparrow \\ N_1}}{0}, 1, 1, \dots) \\
&= F(a_1, a_2, \dots, a_{m-1}, 0, 1, 1, \dots, 1, \underset{\substack{\uparrow \\ N_1}}{1}, 0, 0, \dots).
\end{aligned}
$$

Hence:

$$
F(u_n) - F(\alpha) \geqslant 2^{-m} - \left(2^{-(m+1)} + 2^{-(m+2)} + \dots + 2^{-N_1}\right) = 2^{-N_1} \geqslant 2^{-N}.
$$

(iii) **Case 3:** $a_m = 1$ and $m = k$, where k is, as before, the greatest integer less than N_1 such that $a_k = 1$.



We have

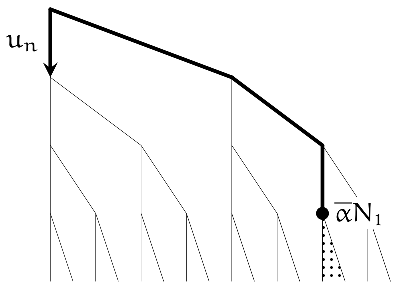
$$
F(\alpha) \geqslant F(\beta) = F(a_1, a_2, \dots, a_{k-1}, 0, 1, 1, \dots).
$$

Since $\overline{\beta}N_2 \in B$ and u_n is not blocked by B,

$$F(u_n) \leqslant F(a_1, a_2, \dots, a_{k-1}, 0, 1, 1, \dots, 1, \underset{\substack{\uparrow \\ N_2}}{0}, 1, 1, \dots);$$

hence $F(\alpha) - F(u_n) \geqslant 2^{-N_2} \geqslant 2^{-N}$.

(iv) **Case 4:** $a_m = 1$ and $m < k$.



We have:

$$\begin{aligned} F(u_n) &\leqslant F(a_1, a_2, \dots, a_{m-1}, 0, 1, 1, \dots) \\ &= F(a_1, a_2, \dots, a_{m-1}, 1, 0, 0, \dots), \\ \text{and } F(\alpha) &\geqslant F(a_1, a_2, \dots, a_{m-1}, 1, 0, 0, \dots, 0, \underset{\substack{\uparrow \\ N_1 - 1}}{1}, 0, 0, \dots); \end{aligned}$$

hence $F(\alpha) - F(u_n) \geqslant 2^{-(N_1-1)} > 2^{-N_1} \geqslant 2^{-N}$.

In all cases, we have $|F(\alpha) - z_n| \geqslant 2^{-N}$, and a similar argument holds when $a_{N_1} = 1$. That is, (z_n) is eventually bounded away from $F(\alpha)$ for each $\alpha \in \mathbf{2}^{\mathbf{N}^+}$. It follows by **Lemma 4.1** that (z_n) is a Specker sequence in $[0, 1]$. □

We now strengthen this result by showing that **anti-FT**$_{\text{Full}}$ and **Speck**$_{[0,1]}$ are in fact equivalent over **BISH**, and furthermore, **anti-FT**$_{\text{c}}$ also falls into their equivalence class [BDMJ12].

Lemma 4.3: BISH $\vdash$ If $(z_n)_{n\geqslant 1}$ is a Specker sequence in $[0, 1]$, then there exists a sequence $(u_n)_{n\geqslant 1}$ in $\mathbf{2}^*$ such that $(F(u_n))_{n\geqslant 1}$ is a Specker sequence, and $|z_n - F(u_n)| < 2^{-n}$ for each $n \in \mathbf{N}^+$.

Proof. Observe that the *dyadic rational numbers* — those of the form $F(u)$ for some $u \in \mathbf{2}^*$ — are dense in $[0, 1]$. Hence we see that for each $n \in \mathbf{N}^+$, there exists $u_n \in \mathbf{2}^*$ with $|z_n - F(u_n)| < 2^{-n}$. Define a sequence $(y_n)_{n \geqslant 1}$ by $y_n = F(u_n)$. Since (z_n) is a Specker sequence we can, given $x \in [0, 1]$, pick $N \in \mathbf{N}^+$ such that $|z_n - x| > 2^{-N+1}$ for all $n \geqslant N$. Then for such n, we have

$$\begin{aligned} \left|y_n - x\right| &\geqslant \left|x - z_n\right| - \left|z_n - y_n\right| \\ &> 2^{-N+1} - 2^{-n} \\ &\geqslant 2^{-N+1} - 2^{-N} = 2^{-N}. \end{aligned}$$

Hence (y_n) is a Specker sequence. □

Proposition 4.4: BISH + Speck$_{[0,1]}$ $\vdash$ anti-FT$_c$.

Proof. Let $(z_n)_{n \geqslant 1}$ be a Specker sequence in $[0, 1]$. In view of **Lemma 4.3**, we may assume that for each $n \in \mathbf{N}^+$ there exists a path $u_n \in \mathbf{2}^*$ such that $z_n = F(u_n)$. Appending zeroes as necessary, we may further assume that $|u_n| \geqslant n$ for each n. Following a proof of Berger and Bridges [BB07, pp. 200–201], we see that

$$D \equiv \left\{ u \in \mathbf{2}^* : \; \left|F(u) - z_{|u|}\right| > 2^{-|u|+1} \right\}$$

is detachable (since the numbers z_n are rational), and that

$$B \equiv \left\{ u \in \mathbf{2}^* : \; (\forall v)\left[u * v \in D\right] \right\}$$

is a c-bar for $\mathbf{2}^*$. Given any $n \in \mathbf{N}^+$, suppose that $\overline{u_n}(n)$ is blocked by B. Then $\overline{u_n}(n) \in B$ (since c-subsets are closed under extensions), and so

$$\left|F(\overline{u_n}(n)) - z_n\right| > 2^{-n+1} > 2^{-n} \geqslant \left|F(\overline{u_n}(n)) - F(u_n)\right| = \left|F(\overline{u_n}(n)) - z_n\right|,$$

which is absurd. Hence $\overline{u_n}(n)$ misses B, and we conclude that B is nonuniform. □

Corollary 4.5: **BISH** $\vdash$ The following are equivalent.

(i) $\mathbf{Speck}_{[0,1]}$.

(ii) **anti-FT**$_{\mathrm{c}}$.

(iii) **anti-FT**$_{\Pi^0_1\mathrm{cl}}$.

(iv) **anti-FT**$_{\Pi^0_1}$.

(v) **anti-FT**$_{\mathrm{Full}}$.

Proof. **Proposition 4.4** shows that (i) $\Longrightarrow$ (ii), and **Proposition 4.2** shows that (v) $\Longrightarrow$ (i). To complete the proof, observe that all c-bars are Π^0_1cl-bars, which are themselves Π^0_1-bars, which are themselves bars; hence we have (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v). □

Contraposing these equivalences, we arrive at our fan-theoretic equivalents of $\mathbf{AS}^{\neg}$; namely, *weak fan theorems* of the form

$\mathbf{FT}^{\neg\neg}_{?}$ For every ?-bar B for $\mathbf{2}^*$, it is impossible that B be nonuniform.

Corollary 4.6: **BISH** $\vdash$ The following are equivalent.

(i) $\mathbf{AS}^{\neg}$.

(ii) $\mathbf{FT}^{\neg\neg}_{\mathrm{c}}$.

(iii) $\mathbf{FT}^{\neg\neg}_{\Pi^0_1\mathrm{cl}}$.

(iv) $\mathbf{FT}^{\neg\neg}_{\Pi^0_1}$.

(v) $\mathbf{FT}^{\neg\neg}_{\mathrm{Full}}$.

Structurally, these fan theorems are similar to — though not the same as[2] — double-negated variations upon $\mathbf{FT}_{\mathrm{c}}$ through $\mathbf{FT}_{\mathrm{Full}}$ (hence the notation). For non-decidable properties, such a double-negation-like construction represents a loss of information: in this case, the lost information corresponds to

[2]Due to our positive characterisation of nonuniform bars.

the structural distinctions between these weak fan theorems. This may be why $\mathbf{FT}_{\Delta}^{\neg\neg}$ seems *not* to fall into the equivalence class of Corollary 4.6: since it refers to a property that is in part decidable, its weakening of $\mathbf{FT}_{\Delta}$ represents a lesser degree of information loss.

4.2 More Principles Equivalent to $\mathbf{Speck}_{[0,1]}$

We now return to the equivalence class of $\mathbf{Speck}_{[0,1]}$, and consider antitheses of: the uniform and uniform sequential continuity theorems from Section 2.4; a boundedness principle; and the following *strong positivity property*, obtained from **POS** by weakening its hypotheses.

$\mathbf{SPOS}_{[0,1]}$ Each pointwise continuous function $f\colon [0,1] \to \mathbf{R}^{+}$ has positive infimum.

Bridges has shown [Bri11b] that $\mathbf{SPOS}_{[0,1]}$ is equivalent to $\mathbf{UCT}_{[0,1],\mathbf{R}}$. It therefore comes as no surprise that its antithesis $\mathbf{anti\text{-}SPOS}_{[0,1]}$ falls into the same equivalence class as $\mathbf{anti\text{-}UCT}_{[0,1],\mathbf{R}}$ (which we shall define shortly). As for $\mathbf{anti\text{-}UCT}_{[0,1],\mathbf{R}}$ itself, recall that $\mathbf{UCT}_{[0,1],\mathbf{R}}$ sits neatly between the fan theorems $\mathbf{FT}_{\Pi^0_1\mathrm{cl}}$ and $\mathbf{FT}_{\mathrm{c}}$ in our greater CRM picture: it is thus similarly unsurprising that its antithesis falls into the same equivalence class as $\mathbf{anti\text{-}FT}_{\Pi^0_1\mathrm{cl}}$ and $\mathbf{anti\text{-}FT}_{\mathrm{c}}$ (recall **Corollary 4.5**).

The purpose of this next result is to give a feeling for just how rich this equivalence class is. While we will later in this chapter identify a further two potentially distinct classes, it would appear that neither contains the same number nor variety of interesting principles as that of $\mathbf{Speck}_{[0,1]}$.[3]

Proposition 4.7: BISH $\vdash$ The following are equivalent.

(i) $\mathbf{Speck}_{[0,1]}$.

(ii) **anti-bdd** There exists a pointwise continuous function $f\colon [0,1] \to \mathbf{R}$ that is unbounded above in the sense that, for each $n \in \mathbf{N}$, there exists $x \in [0,1]$ with $f(x) > n$.

[3] Note, though, that **Proposition 4.5.1** of [Die08, pp. 65–67] establishes some equivalents of $\mathbf{anti\text{-}FT}_{\Delta}$ not covered here.

(iii) $\mathbf{anti\text{-}UCT}_{[0,1],\mathbf{R}}$ There exists a pointwise continuous function $f\colon [0,1] \to \mathbf{R}$ that is non-uniformly-continuous in the following sense: there exists $\epsilon > 0$ such that for each $\delta > 0$, there exist $x, x' \in [0,1]$ with $|x - x'| < \delta$ and $|f(x) - f(x')| > \epsilon$.

(iv) $\mathbf{anti\text{-}USCT}_{[0,1],\mathbf{R}}$ There exists a pointwise continuous function $f\colon [0,1] \to \mathbf{R}$ that is non-uniformly-sequentially-continuous in the following sense: there exist $\epsilon > 0$ and sequences $(x_n)_{n\geqslant 1}$ and $(x'_n)_{n\geqslant 1}$ in $[0,1]$ with $|x_n - x'_n| \to 0$, such that $|f(x_n) - f(x'_n)| \geqslant \epsilon$ for each n.

(v) $\mathbf{anti\text{-}SPOS}_{[0,1]}$ There exists a pointwise continuous function $f\colon [0,1] \to \mathbf{R}^+$ with infimum 0.

Proof. The equivalence of $\mathbf{Speck}_{[0,1]}$, **anti-bdd** and $\mathbf{anti\text{-}UCT}_{[0,1],\mathbf{R}}$ is proved in [Die08, pp. 67–70]. One can see that $\mathbf{anti\text{-}SPOS}_{[0,1]}$ also falls into this equivalence class by observing that it is equivalent to **anti-bdd**: if $f\colon [0,1] \to \mathbf{R}^+$ is a pointwise continuous function with infimum 0, then the reciprocal $1 / f$ is a witness[4] to **anti-bdd**. Conversely, if $f\colon [0,1] \to \mathbf{R}$ is a pointwise continuous function that is unbounded above, then $1 / \max\{1, f\}$ is a witness to $\mathbf{anti\text{-}SPOS}_{[0,1]}$.

It remains to consider $\mathbf{anti\text{-}USCT}_{[0,1],\mathbf{R}}$. Bridges has shown [Bri09b, p. 443] that $\mathbf{USCT}_{[0,1],\mathbf{R}}$ entails $\mathbf{AS}_{[0,1]}$, and his construction may easily be adapted to prove

$$\mathbf{BISH} + \mathbf{Speck}_{[0,1]} \vdash \mathbf{anti\text{-}USCT}_{[0,1],\mathbf{R}}.$$

To complete the proof, we need only observe that $\mathbf{anti\text{-}UCT}_{[0,1],\mathbf{R}}$ is an immediate consequence of $\mathbf{anti\text{-}USCT}_{[0,1],\mathbf{R}}$. □

It is interesting, when examining antitheses such as $\mathbf{anti\text{-}SPOS}_{[0,1]}$, to consider the assumptions under which one could pass from (in this case) $\neg\,\mathbf{anti\text{-}SPOS}_{[0,1]}$ back to $\mathbf{SPOS}_{[0,1]}$. In doing so, one would (in light of the implications of **Figure 4.2**) collapse several significant principles into a single equivalence class. At a glance, it seems as though Markov's principle might be enough to accomplish this, by the following sort of argument:

[4] A *witness* to an existential statement $\phi \equiv (\exists x \in X)P(x)$ is an object $t \in X$ for which $P(t)$ holds. Constructing such a witness thus demonstrates the truth of ϕ.

Let $f\colon [0,1] \to \mathbf{R}^+$ be a pointwise continuous function. If we had $\inf f = 0$, then $\textbf{anti-SPOS}_{[0,1]}$ would hold; hence $\neg(\inf f = 0)$. Furthermore, since $0 < f(x)$ for all $x \in [0,1]$, we have $\neg(\inf f < 0)$; we therefore see that $\neg(\inf f \leqslant 0)$. But **MP** is equivalent to the claim that $\neg(x \leqslant 0)$ entails $x > 0$ for each real number x [Ish04, p. 5]. Hence we have $\inf f > 0$.

$$
\begin{tikzcd}
\mathbf{UCT}_{[0,1],\mathbf{R}} \arrow[rr, leftrightarrow] \arrow[d] & & \mathbf{SPOS}_{[0,1]} \\
\mathbf{FT}_c \arrow[r, leftrightarrow] & \mathbf{AS}_{[0,1]} \arrow[d] & \\
& \neg\,\mathbf{Speck}_{[0,1]} \arrow[r, leftrightarrow] & \neg\,\textbf{anti-SPOS}_{[0,1]} \arrow[uu, dashed, "?"']
\end{tikzcd}
$$

Figure 4.2: Implications between $\neg\,\textbf{anti-SPOS}_{[0,1]}$ and $\mathbf{SPOS}_{[0,1]}$

The problem here is that this argument assumes the existence of $\inf f$, which is by no means guaranteed, even in the presence of **MP**. To illustrate this, we show that such an assumption is provably false in **RUSS**. In doing so, we will make use of the following definition: for each $t \in \mathbf{R}$ and $\delta > 0$, the *spike function* $s(t, \delta, \cdot)\colon \mathbf{R} \to [0,1]$ is the unique uniformly continuous function with the following properties [Bri09b, p. 441]:

- $s(t, \delta, t) = 1$;
- $s(t, \delta, x) = 0$ whenever $|x - t| > \delta$; and
- $s(t, \delta, \cdot)$ is linear in $[t - \delta, t]$ and $[t, t + \delta]$.

By the *support* of a spike function, we refer to the region where it is nonzero.

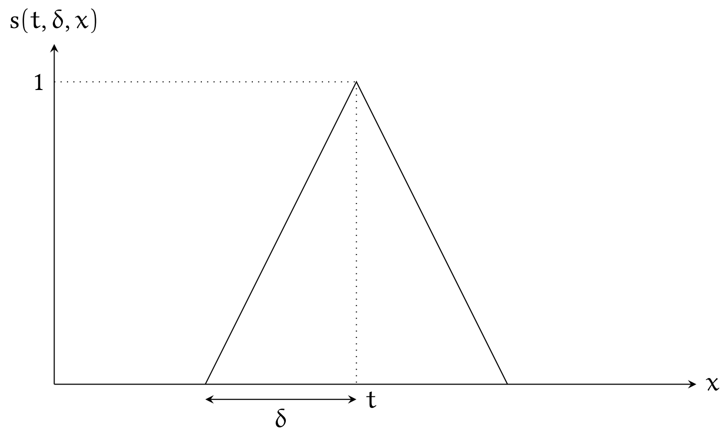


Figure 4.3: The spike function $s(t, \delta, \cdot)$

We now give our intended result [BDMJ12].

Proposition 4.8: **RUSS** $\vdash$ The following statement is false.

($*$) Every pointwise continuous function $g\colon [0,1] \to \mathbf{R}^+$ has an infimum.

Proof. Begin by observing that, within **RUSS**, Specker's theorem (**Proposition 2.2**) allows us to easily construct *strictly* increasing Specker sequences. To do so, take a nondecreasing Specker sequence $(x_n)_{n\geqslant 1}$, and make the following inductive construction: let $n_1 = 1$. Then, having found n_i for some $i \geqslant 1$, use the fact that (x_n) is eventually bounded away from x_{n_i} to find $n_{i+1} > n_i$ such that $x_{n_{i+1}} \neq x_{n_i}$ (and therefore $x_{n_{i+1}} > x_{n_i}$). The subsequence $(x_{n_k})_{k\geqslant 1}$ thus defined is strictly increasing, and inherits the Specker property from (x_n).

With this in mind, let $(z_n)_{n\geqslant 1}$ be a strictly increasing Specker sequence in $[0,1]$ [BR87, pp. 58–59]. Construct a sequence $(\delta_n)_{n\geqslant 1}$ of positive numbers such that $\delta_1 < z_1$ and for each n, both $z_n + \delta_n < z_{n+1} - \delta_{n+1}$ and $\delta_n < 2^{-n}$ hold. (This ensures that the supports of the spike functions we will soon employ are disjoint.) Fix any binary sequence $(\lambda_n)_{n\geqslant 1}$, and define a sequence $(f_n)_{n\geqslant 1}$ of uniformly continuous functions on $[0,1]$ as follows:

- if $\lambda_n = 0$, then set $f_n = 0$; and
- if $\lambda_n = 1$, then set $f_n = s(z_n, \delta_n, \cdot)$.

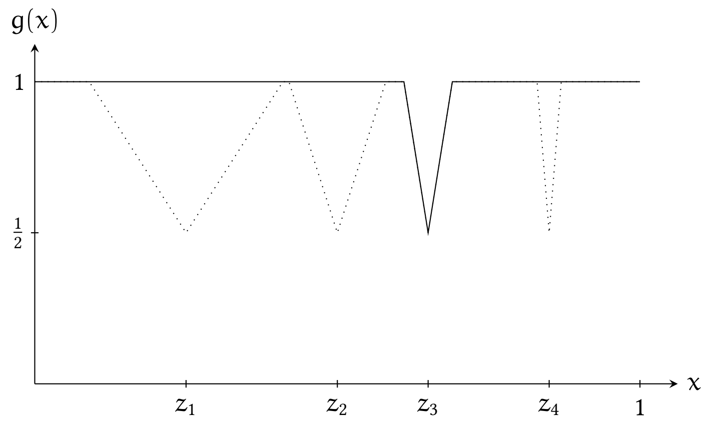


Figure 4.4: The function g from the proof of **Proposition 4.8**, in the case where $(\lambda_n) = (0, 0, 1, 0, \dots)$

For each $x \in [0, 1]$, since $(z_n)_{n \geqslant 1}$ is eventually bounded away from x, there exists $N_x \in \mathbf{N}^+$ such that $|x - z_n| > 2^{-N_x}$ for all $n \geqslant N_x$. Then, $f_n(x) = 0$ for all $n \geqslant N_x$, so that

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{N_x - 1} f_n(x). \tag{4.1}$$

Hence the function $g \equiv 1 - \frac{1}{2} \sum_{n=1}^{\infty} f_n$ (depicted in **Figure 4.4**) is well-defined, pointwise continuous and positive-valued on $[0, 1]$. Assuming $(*)$, then, we can find the infimum $\mu = \inf g$, whence two possible cases arise.

- If $\mu > \frac{1}{2}$, then immediately, $\lambda_n = 0$ for all n.
- If $\mu < 1$, then we can compute a value x for which $g(x) < 1$. We then see that $\sum_{n=1}^{\infty} f_n(x) > 0$, whence by (4.1) there exists k such that $f_k(x) > 0$ and therefore $\lambda_k = 1$.

Hence the statement $(*)$ implies **LPO**, which is provably false in **RUSS** [BR87, p. 53]. □

4.3 Principles Stronger Than $\textbf{Speck}_{[0,1]}$

While all of the recursive principles examined so far in this chapter have been equivalent to $\textbf{Speck}_{[0,1]}$, there remain three more to consider that fall into potentially stronger equivalence classes. The first of these is an antithesis to the positivity property, of the general form:

anti-POS$_X$ There exists a uniformly continuous function $f\colon X \to \mathbf{R}^+$ with infimum 0,

for some metric space X. We will be most interested in the case where $X = [0,1]$: just as **POS** and $\textbf{FT}_\Delta$ are equivalent, we will see that $\textbf{anti-POS}_{[0,1]}$ and $\textbf{anti-FT}_\Delta$ are too. Furthermore, we simultaneously show that three different formulations of this anti-positivity property are equivalent, in much the same pattern as that of **Proposition 3.7**.

Proposition 4.9: $\textbf{BISH} + \textbf{anti-POS}_{2^{\mathbf{N}^+}} \vdash \textbf{anti-FT}_\Delta$.

Proof. Let $f\colon \mathbf{2}^{\mathbf{N}^+} \to \mathbf{R}^+$ be a uniformly continuous function with infimum 0, and identify $f(u)$ with $f(u * 000\dots)$ for each $u \in \mathbf{2}^*$. We proceed by reprising a construction of Berger and Bridges [BB08a, pp. 133–134]. Since $\rho(\alpha, \overline{\alpha}n) \leqslant 2^{-n}$ for every $\alpha \in \mathbf{2}^{\mathbf{N}^+}$ and $n \in \mathbf{N}$, the uniform continuity of f allows us to construct, using countable choice, a strictly increasing sequence $(n_k)_{n \geqslant 1}$ of positive integers such that for each $\alpha \in \mathbf{2}^{\mathbf{N}^+}$ and $k \in \mathbf{N}^+$,

$$\left|f(\alpha) - f(\overline{\alpha}n_k)\right| < 2^{-k}.$$

Again following [BB08a, p. 133], we use countable choice to construct a binary family $(\lambda_u)_{u \in \mathbf{2}^*}$ such that for each $u \in \mathbf{2}^*$,

$$\lambda_u = 0 \implies (\forall k)\left[|u| \neq n_k\right] \vee (\exists k)\left[|u| = n_k \wedge f(u) < 2^{-k+2}\right], \text{ and}$$
$$\lambda_u = 1 \implies (\exists k)\left[|u| = n_k \wedge f(u) > 2^{-k+1}\right].$$

Then

$$B \equiv \left\{u \in \mathbf{2}^* \colon \lambda_u = 1\right\}$$

is a detachable subset of $\mathbf{2}^*$, and the argument of [BB08a, pp. 133–134] shows that B is a bar. We now use a reversal of the final part of this argument to see that B is nonuniform. Since $\inf f = 0$ we can, given any positive integer k, choose an infinite path α such that $f(\alpha) < 2^{-k}$. We then have

$$\begin{aligned} f(\overline{\alpha}n_k) &\leqslant \left|f(\alpha) - f(\overline{\alpha}n_k)\right| + f(\alpha) \\ &< 2^{-k} + 2^{-k} = 2^{-k+1}, \end{aligned}$$

so $\lambda_{\overline{\alpha}n_k} = 0$ and therefore $\overline{\alpha}n_k \notin B$. Since k is arbitrary and for each positive integer n there exists k with $n_k > n$, it follows that B is nonuniform. $\square$

Proposition 4.10: $\mathbf{BISH} + \textbf{anti-FT}_\Delta \vdash \textbf{anti-POS}_{[0,1]}$.

Proof. Let B be a detachable bar for $\mathbf{2}^*$. The Julian-Richman result (**Proposition 3.6**) states that there exists a uniformly continuous function $f\colon [0,1] \to \mathbf{R}^+$ such that $\inf f > 0$ if and only if B is a uniform bar for $\mathbf{2}^*$. So if B is nonuniform, we must have $\inf f = 0$ (remember that the uniform continuity of f ensures that $\inf f$ exists). $\square$

Corollary 4.11: $\mathbf{BISH} \vdash$ The following are equivalent.

(i) $\textbf{anti-POS}_{\mathbf{2}^{\mathbf{N}^+}}$.

(ii) $\textbf{anti-FT}_\Delta$.

(iii) $\textbf{anti-POS}_{[0,1]}$.

(iv) $\textbf{anti-POS}_X$ for some compact metric space X.

Proof. **Proposition 4.9** and **Proposition 4.10** establish that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii), and (iii) $\Longrightarrow$ (iv) is trivial. To see that (iv) $\Longrightarrow$ (i), let $f\colon X \to \mathbf{R}^+$ be the uniformly function with $\inf f = 0$ given by $\textbf{anti-POS}_X$. We again use **Theorem 1.4** of [BR87, p. 106], as we did throughout Chapter 3, to construct a uniformly continuous surjective function $g\colon \mathbf{2}^{\mathbf{N}^+} \to X$. Then as in **Proposition 3.7**, $\inf f \circ g = \inf f$, so $f \circ g$ is our required witness to anti-positivity on $\mathbf{2}^{\mathbf{N}^+}$. $\square$

In light of this result, we denote the anti-positivity properties in this equivalence class — (iii) in particular — by simply **anti-POS**.

Our final observation about $\textbf{anti-FT}_\Delta$ and **anti-POS** links them to the class of principles equivalent to $\textbf{Speck}_{[0,1]}$ via the following *increasing Specker property*:

$\mathbf{Speck}^{\uparrow}_{[0,1]}$ There exists a *nondecreasing* sequence in $[0,1]$ that is eventually bounded away from each point of $[0,1]$.

Proposition 4.12: $\mathbf{BISH} + \mathbf{anti\text{-}FT}_{\Delta} \vdash \mathbf{Speck}^{\uparrow}_{[0,1]}$.

Proof. Let B be a detachable bar that is nonuniform, and for each $n \in \mathbf{N}$, let u_n be the leftmost path of length n that is not blocked by B. (The detachability of B allows us to find this path.) A proof of Diener [Die08, p. 68] shows that the sequence $(z_n) \equiv (G(u_n))$ in $[0,1]$ is eventually bounded away from each point of $[0,1]$, where G is our embedding of paths into the Cantor set from Section 3.1. We show furthermore that (z_n) is nondecreasing.

Fix any $n \in \mathbf{N}^+$, and let $k \leqslant n$ be the least number for which $[u_{n+1}]_k \neq [u_n]_k$. (If no such k exists, then $\overline{u_{n+1}}(n) = u_n$, so $G(u_{n+1}) \geqslant G(u_n)$ and we have nothing more to prove.) If $[u_{n+1}]_k = 0$ and $[u_n]_k = 1$, then $\overline{u_{n+1}}(n)$ is a path of length n left of u_n that misses B. This contradicts the construction of u_n; hence we must have $[u_{n+1}]_k = 1$ and $[u_n]_k = 0$. It then follows that $G(u_{n+1}) > G(u_n)$.

Hence (z_n) is a nondecreasing Specker sequence in $[0,1]$, as required. □

Over on the intuitionistic side of things, this gives us another weak-fan-theoretic result, this time pertaining to the *increasing* non-Specker property. It is not yet known whether this implication can be reversed.

Corollary 4.13: $\mathbf{BISH} + \mathbf{AS}^{\uparrow\neg}_{[0,1]} \vdash \mathbf{FT}^{\neg\neg}_{\Delta}$.

Figure 4.5 gives a summary of the recursive results of this chapter. Several open problems remain. The first is the matter of whether the implication from $\mathbf{anti\text{-}FT}_{\Delta}$ and **anti-POS** to $\mathbf{Speck}^{\uparrow}_{[0,1]}$ could be reversed. If so, this would give a rather satisfying partitioning of all the antitheses we have identified, especially if we could also give a proof of inequivalence to properly separate the resulting equivalence classes for $\mathbf{Speck}_{[0,1]}$ and $\mathbf{Speck}^{\uparrow}_{[0,1]}$.

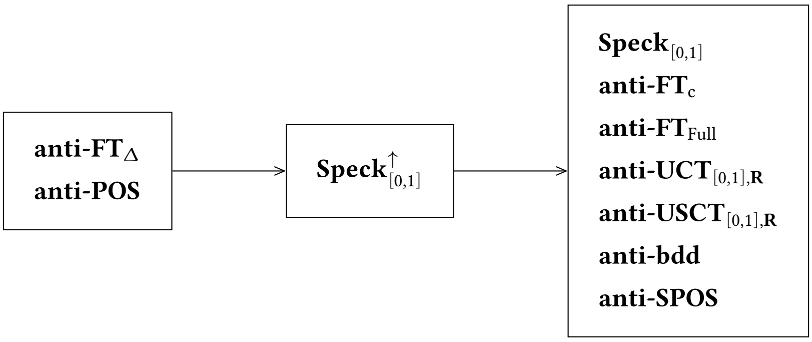


Figure 4.5: Relationships between antitheses of important principles, relative to **BISH**

However, it may be the case that the existence of any Specker sequence at all implies the existence of a nondecreasing one. The results obtained here indicate that, as an alternative to attempting a direct proof, we could aim to show that (say)

$$\mathbf{BISH} + \mathbf{Speck}_{[0,1]} \ \vdash \ \textbf{anti-POS},$$

thereby collapsing all of our antitheses into a single equivalence class. This seems unlikely, but were it possible, we would — in the spirit of constructive mathematics — be able to obtain an algorithm for generating a nondecreasing Specker sequence, given some unordered one.

Coming up with such an algorithm directly is similarly difficult. A naïve strategy would be to fix a Specker sequence (z_n), and aim to somehow use the fact that it is eventually bounded away from each of its own terms to construct a nondecreasing subsequence (which would then automatically inherit the Specker property from (z_n)). However, the following Brouwerian counterexample rules out this approach.

Proposition 4.14: The following statement is nonconstructive.

($*$) If $(z_n)_{n\geqslant 1}$ is a Specker sequence, then (z_n) has a monotone subsequence.

Proof. Were ($*$) constructively valid, it would hold within **RUSS**. Working in **RUSS**, then: fix any binary sequence $(\lambda_n)_{n\geqslant 1}$, and let $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ be Specker sequences in $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$, respectively, such that (x_n) is strictly increasing and (y_n) is strictly decreasing.[5] Now define a sequence $(z_n)_{n \geqslant 1}$ as follows:

$$z_n = \begin{cases} x_n & \text{if } \lambda_m = 0 \text{ for all } m \leqslant n, \text{ and} \\ y_n & \text{if } \lambda_m = 1 \text{ for some } m \leqslant n. \end{cases}$$

Clearly, (z_n) is a Specker sequence. For, since (x_n) and (y_n) are both eventually bounded away from $\frac{1}{2}$, we can find $\delta > 0$ such that $x_n < \frac{1}{2} - \delta$ and $y_n > \frac{1}{2} + \delta$ for all n. Then, given any $x \in [0, 1]$, either $x \in \left[0, \frac{1}{2} - \frac{1}{2}\delta\right)$, $x \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right)$ or $x \in \left(\frac{1}{2} + \frac{1}{2}\delta, 1\right]$; in any case, (x_n) and (y_n) are both eventually bounded away from x, so (z_n) is also.

Now, assume that $(*)$ holds, and find a monotone subsequence $(z_{n_k})_{k \geqslant 1}$ of (z_n). Since the terms of (z_n) are distinct, this means that either

- (z_{n_k}) is strictly increasing, whence

  $$(\forall k)(\exists i \geqslant k)\left[z_{n_k} = x_i\right]$$

  and $\lambda_m = 0$ for all m; or

- (z_{n_k}) is strictly decreasing, whence

  $$(\forall k)(\exists i \geqslant k)\left[z_{n_k} = y_i\right]$$

  and $\lambda_m = 1$ for some m.

Hence we have derived **LPO**, which is provably false within **RUSS**. So $(*)$ cannot hold constructively. $\square$

So any algorithm for constructing a nondecreasing Specker sequence from an unordered one must do something more ingenious than merely taking a subsequence (and will have to demonstrate from scratch that the constructed sequence is indeed eventually bounded away from each point of $[0, 1]$).

[5] Recall the construction from the proof of **Proposition 4.8**.

If, however, it is the case that no such algorithm exists, the relevant problem becomes instead that of finding

- a model of **BISH** separating $\mathbf{Speck}_{[0,1]}$ and $\mathbf{Speck}^{\uparrow}_{[0,1]}$, and
- a principle whose addition to **BISH** would enable us to pass from $\mathbf{Speck}_{[0,1]}$ to $\mathbf{Speck}^{\uparrow}_{[0,1]}$ (and, correspondingly, from $\mathbf{AS}^{\uparrow\neg}_{[0,1]}$ to $\mathbf{AS}^{\neg}$ and $\mathbf{AS}^{\uparrow}_{[0,1]}$ to $\mathbf{AS}^{\mathbf{ltd}}$).

We conclude this part of the thesis by summarising in **Figure 4.6** the key anti-Specker relationships from these last two chapters. Note also that in the presence of Markov's principle, one can easily verify the following equivalences, and thus obtain the equivalence classes of **Figure 4.7**.

Proposition 4.15: $\mathbf{BISH} + \mathbf{MP}\ \vdash$

(i) $\mathbf{AS}^{\mathbf{ltd}} \iff \mathbf{AS}^{\neg}$;

(ii) $\mathbf{AS}^{\uparrow} \iff \mathbf{AS}^{\uparrow\neg}$; and

(iii) $\mathbf{FT}_{\Delta} \iff \mathbf{FT}^{\neg\neg}_{\Delta}$.

The system $\mathbf{BISH} + \mathbf{MP}$ is one in which it is permissible to perform a specific type of double negation elimination: namely, if P is a decidable predicate on $\mathbf{N}$, one may deduce

$$\frac{\neg\neg(\exists n \in \mathbf{N})\mathrm{P}(n)}{\therefore\ (\exists n \in \mathbf{N})\mathrm{P}(n).}$$

Each of the equivalences of **Proposition 4.15** relates a principle of this first (double-negated) form to one of the second form. For (iii), this relationship is easy to see; for (i), one has to bear in mind the alternative characterisation of $\mathbf{AS}^{\neg}$ given on page 35. (A similar formulation can be made for (ii).)

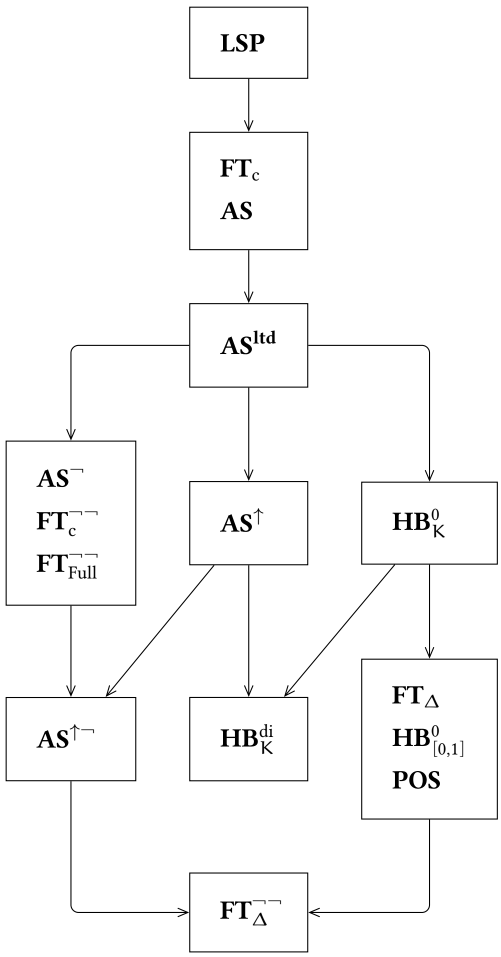


Figure 4.6: Relationships between anti-Specker properties, relative to **BISH**

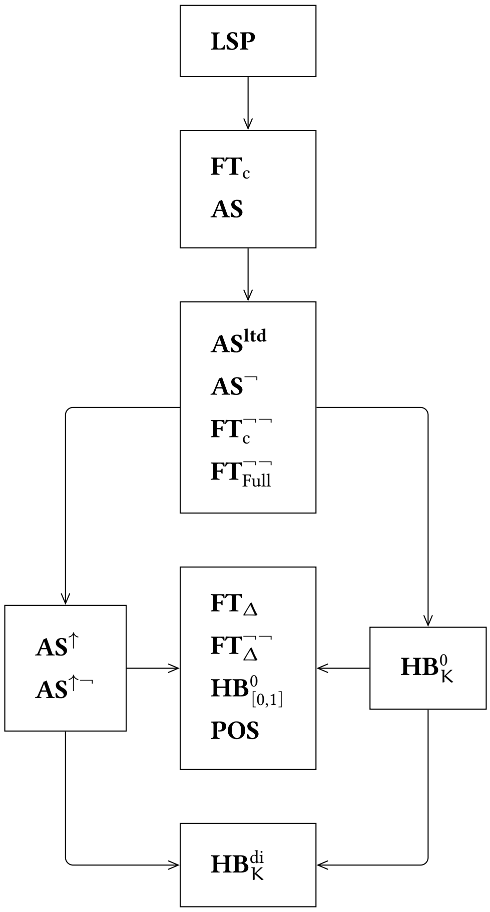


Figure 4.7: Relationships between anti-Specker properties, relative to $\mathbf{BISH} + \mathbf{MP}$

5 | Interlude: Omniscience and Fan Theorems

While the various omniscience principles are reasonably well-understood in the context of constructive reverse mathematics, little work has been done to *directly* explore their connections with the fan theorems. In this chapter we put aside our anti-Specker properties for a little, and investigate some of these connections in as direct a manner as possible. The proof techniques we will employ include the consideration of *complete subfans* of $\mathbf{2}^*$, each of which we denote by

$$\mathbf{csf}(u) \equiv \{u * v\colon\ v \in \mathbf{2}^*\}$$

for $u \in \mathbf{2}^*$. The first connection we will examine is the following.

Proposition 5.1: $\mathbf{BISH} + \mathbf{LLPO} \vdash \mathbf{FT}_\Delta$.

This result was originally established by Ishihara in [Ish90], by an indirect route. He showed that, over **BISH**, **LLPO** is equivalent to the *minimum principle* **MIN**, then observed that **MIN** entails **POS** (which is in turn equivalent to $\mathbf{FT}_\Delta$). His later article [Ish06] presents another approach: it proves directly that the *weak König lemma* **WKL** — which is also equivalent to **LLPO** [Ish90] — implies $\mathbf{FT}_\Delta$.[1]

However, to better understand the relationship between $\mathbf{FT}_\Delta$ and **LLPO** *itself*, we now give two fully direct proofs of **Proposition 5.1**, published in [BDMJ13b]. The first proceeds by a tree-halving argument and makes use of the following lemma.

[1]In fact, Diener [Die13] has recently proved that **WKL** entails the uniform continuity theorem $\mathbf{UCT}_{[0,1],\mathbf{R}}$.

Lemma 5.2: **BISH + LLPO** $\vdash$ If B is a detachable bar for $\mathbf{2}^*$, then it is impossible that, for each $n \in \mathbf{N}$, there is a finite path of length n that misses B.

Proof. Let B be a detachable bar for $\mathbf{2}^*$, and suppose that, for each $n \in \mathbf{N}$, there exists a finite path of length n that misses B. We inductively construct an infinite path $\alpha \equiv (a_n)_{n \geqslant 1}$ in $\mathbf{2}^*$ such that, for each n:

(∗) For every $m \geqslant n$, there exists a path of length m in $\mathbf{csf}(\overline{\alpha}n)$ that misses B.

(In particular, this means that $\overline{\alpha}n \notin B$.) Start the induction by observing that, by assumption, the empty path λ satisfies property (∗). Now suppose we have constructed the first k terms of α so that $\overline{\alpha}k$ satisfies (∗). Denote the left and right halves of $\mathbf{csf}(\overline{\alpha}k)$ by $H_L \equiv \mathbf{csf}(\overline{\alpha}k * 0)$ and $H_R \equiv \mathbf{csf}(\overline{\alpha}k * 1)$, respectively, and define a binary sequence $(\lambda_n)_{n \geqslant 1}$ as follows:

- If n is odd, set $\ell = k + \frac{1}{2}(n+1)$. Then:
  - if some prior odd term of (λ_n) is equal to 1, or there exists a path in H_L of length ℓ that misses B, then set $\lambda_n = 0$; and
  - if all prior odd terms of (λ_n) are equal to 0, and there exists a path in H_L of length $(\ell - 1)$ that misses B but all paths in H_L of length ℓ are blocked by B, then set $\lambda_n = 1$.
- If n is even, set $\ell = k + \frac{1}{2}n$. Then:
  - if some prior even term of (λ_n) is equal to 1, or there exists a path in H_R of length ℓ that misses B, then set $\lambda_n = 0$; and
  - if all prior even terms of (λ_n) are equal to 0, and there exists a path in H_R of length $(\ell - 1)$ that misses B but all paths in H_R of length ℓ are blocked by B, then set $\lambda_n = 1$.

Each successive odd/even pair of terms of (λ_n) examines all the paths in $\mathbf{csf}(\overline{\alpha}k)$ of length ℓ and reflects which — if either — of its halves is the first to block all paths of this length.

Note that if $\lambda_{2m-1} = 1$, then $\lambda_{2i} = 0$ for all $i \geqslant m$: suppose otherwise that $\lambda_{2i} = 1$ for some $i \geqslant m$. Then by definition, every path in H_R of length $(k+i)$ is blocked by B. But since $\lambda_{2m-1} = 1$, we also have every path in H_L of length $(k+m)$ blocked by B. It follows that all paths in $\mathbf{csf}(\overline{\alpha}k)$

of length $(k+i)$ are blocked by B, which contradicts the induction hypothesis $(*)$. By a similar argument, if $\lambda_{2m} = 1$, then $\lambda_{2i-1} = 0$ for all $i \geqslant m$. Hence (λ_n) contains at most one term equal to 1.

We can now apply **LLPO**. If all the odd terms of (λ_n) are equal to 0, then for each $m \geqslant k+1$, there exists a path of length m in H_L that misses B; hence we satisfy $(*)$ by taking $a_{k+1} = 0$. Similarly, if all the even terms of (λ_n) are 0, we set $a_{k+1} = 1$. This completes our inductive construction of α. But the path so constructed misses B, a contradiction: hence we conclude that it is impossible for there to be finite paths that miss B of every length. $\square$

We are now able to give our first proof of **Proposition 5.1**:

Proof. Let B be a detachable bar for $\mathbf{2}^*$. We construct, inductively, binary sequences $\boldsymbol{\lambda}^{(k)}$ for $k \geqslant 1$, and an infinite path $\alpha \equiv (a_n)_{n \geqslant 1}$, with the following property:

$(*)$ If there exists $m \geqslant k$ such that $\lambda^{(k)}_{2m+1} + \lambda^{(k)}_{2m+2} = 1$, then

- ❖ there exists a path of length m that misses B;
- ❖ all paths of length $(m+1)$ are blocked by B; and
- ❖ the leftmost path w of length $(m+1)$ with $\overline{w}m$ missing B satisfies $\overline{w}k = \overline{\alpha}k$.

Fix $k \geqslant 0$ and suppose that we have constructed the first k terms of α and, if $k \geqslant 1$, the sequence $\boldsymbol{\lambda}^{(k)}$, such that the property $(*)$ is satisfied. Abbreviate the complete subfan $\mathbf{csf}(\overline{\alpha}k)$ by C. We construct the binary sequence $\boldsymbol{\lambda}^{(k+1)}$ as follows. If all paths of length k are blocked by B, take $\boldsymbol{\lambda}^{(k+1)} = \mathbf{0}$ and $a_{k+1} = 0$. Otherwise, we define $\boldsymbol{\lambda}^{(k+1)}$ in a pairwise manner. Given $n \in \mathbf{N}$, we have two cases to deal with:

- ❖ **Case 1:** All paths of length n are blocked by B. We then set $\lambda^{(k+1)}_{2n+1} = \lambda^{(k+1)}_{2n+2} = 0$.
- ❖ **Case 2:** There is a path of length n that misses B. We can then decide between the following subcases:
  - ◆ If there is also a path of length $(n+1)$ that misses B, set $\lambda^{(k+1)}_{2n+1} = \lambda^{(k+1)}_{2n+2} = 0$.
  - ◆ If all paths of length $(n+1)$ are blocked by B, then we will set $\lambda^{(k+1)}_{2n+1}$ and $\lambda^{(k+1)}_{2n+2}$ so that $\lambda^{(k+1)}_{2n+1} + \lambda^{(k+1)}_{2n+2} = 1$. Let w be the leftmost path of length $(n+1)$ with $\overline{w}n$ missing B: we show that C contains w. If $k = 0$, then $C = \mathbf{2}^*$, so this is immediate;

otherwise, consider $\boldsymbol{\lambda}^{(k)}$. Since $\lambda^{(k)}_{2n+1}$ and $\lambda^{(k)}_{2n+2}$ were defined in this same way, we have $\lambda^{(k)}_{2n+1} + \lambda^{(k)}_{2n+2} = 1$; furthermore, since not all paths of length k are blocked by B, we must have $k \leqslant n$. So by property $(*)$, we see that w satisfies $\overline{w}k = \overline{\alpha}k$, whence $w \in C$. Moreover, $|w| = n + 1 \geqslant k + 1$, so w is in fact contained in one of the halves of C:

- if $w \in \mathbf{csf}(\overline{\alpha}k * 0)$, set $\lambda^{(k+1)}_{2n+1} = 1$ and $\lambda^{(k+1)}_{2n+2} = 0$;
- if $w \in \mathbf{csf}(\overline{\alpha}k * 1)$, set $\lambda^{(k+1)}_{2n+1} = 0$ and $\lambda^{(k+1)}_{2n+2} = 1$.

With this construction, we search through $\mathbf{2}^*$ for a longest path that misses B, and set an odd or even term of $\boldsymbol{\lambda}^{(k+1)}$ equal to 1 if this path is in the left or right half of C, respectively. Clearly, $\boldsymbol{\lambda}^{(k+1)}$ has at most one term equal to 1. Hence we can apply **LLPO**:

❖ if $\lambda^{(k+1)}_{2n} = 0$ for all n, set $a_{k+1} = 0$;

❖ if $\lambda^{(k+1)}_{2n-1} = 0$ for all n, set $a_{k+1} = 1$.

This completes the inductive construction of $\boldsymbol{\lambda}^{(k+1)}$ and a_{k+1}. We now prove that the property $(*)$ holds. Suppose that there exists $m \geqslant k+1$ such that $\lambda^{(k+1)}_{2m+1} + \lambda^{(k+1)}_{2m+2} = 1$. Consider the case when $a_{k+1} = 0$ (a similar argument applies when $a_{k+1} = 1$). By the construction of a_{k+1}, we know that all the even terms of $\boldsymbol{\lambda}^{(k+1)}$ are zero, and in particular, $\lambda^{(k+1)}_{2m+2} = 0$. Hence we have $\lambda^{(k+1)}_{2m+1} = 1$. By definition, this tells us that:

❖ there is a path of length m that misses B;

❖ all paths of length $(m + 1)$ are blocked by B; and

❖ if w is the leftmost path of length $(m + 1)$ with $\overline{w}m$ missing B, then $w \in \mathbf{csf}(\overline{\alpha}k * 0)$. That is, $\overline{w}(k + 1) = \overline{\alpha}k * 0$, and since $a_{k+1} = 0$, we have $\overline{w}(k + 1) = \overline{\alpha}(k + 1)$.

So our inductive construction meets its specifications.

Now, since B is a bar, there exists $N \in \mathbf{N}$ such that $\overline{\alpha}N \in B$. Suppose that for some $m \geqslant N$, there exists a path of length m that misses B, but all paths of length $(m + 1)$ are blocked by B. Then $\lambda^{(N)}_{2m+1} + \lambda^{(N)}_{2m+2} = 1$, and so by property $(*)$, the leftmost path w of length $(m + 1)$ with $\overline{w}m$ missing B satisfies $\overline{w}N = \overline{\alpha}N$. Hence $\overline{w}N \in B$, which is absurd since $N \leqslant m$ and $\overline{w}m$ misses B.

This contradiction tells us, for each $m \geqslant N$: if there exists a path of length m that misses B, then not all paths of length $(m+1)$ are blocked by B, and so at least one misses B. A simple induction argument now shows that, if there exists a path of length N that misses B, then there exist paths that miss B of every length. But this contradicts **Lemma 5.2**. Since B is detachable, we conclude that all paths of length N are blocked by B. $\square$

Next we present our second proof of **Proposition 5.1**. We will make use of the following predicate for finite paths in $\mathbf{2}^*$:

$$\mathbf{StartsWith}(u, v) \equiv \begin{cases} \overline{u}(|v|) = v & \text{if } |u| \geqslant |v|; \\ u = \overline{v}(|u|) & \text{if } |u| < |v|. \end{cases}$$

Proof. Again let B be a detachable bar for $\mathbf{2}^*$. We construct, inductively, an infinite path $\alpha \equiv (a_n)_{n \geqslant 1}$ in such a way as to maximise the number N for which some restriction $\overline{\alpha}N$ is first blocked by B. Suppose we have found the first k terms of α, for some $k \geqslant 0$. Define, for $d \in \mathbf{2}$ and $k, n \in \mathbf{N}$, a decidable predicate $\mathbf{Bl}_d^k(n)$ that depends upon these terms:

$$\mathbf{Bl}_d^k(n) \equiv (\forall u \colon |u| = n \wedge \mathbf{StartsWith}(u, \overline{\alpha}k * d))(\exists m \leqslant n)\big[\overline{u}m \in B\big],$$

$$\text{and so } \neg\mathbf{Bl}_d^k(n) \equiv (\exists u \colon |u| = n \wedge \mathbf{StartsWith}(u, \overline{\alpha}k * d))(\forall m \leqslant n)\big[\overline{u}m \notin B\big].$$

For $n > k$, $\mathbf{Bl}_d^k(n)$ asserts that the left (when $d = 0$) or right (when $d = 1$) half of the complete subfan $\mathbf{csf}(\overline{\alpha}k)$ is uniformly blocked by B at depth n. Our use of the robust predicate **StartsWith** extends this sensibly to the case $n \leqslant k$, whence $\mathbf{Bl}_d^k(n)$ becomes equivalent to

$$(\exists m \leqslant n)\big[\overline{\alpha}m \in B\big].$$

(While we do not make use of this case in the proof at hand, we shall find it useful when we reprise this definition for our proof of **Proposition 5.4**.) Notice that $\mathbf{Bl}_d^k(n)$ entails $\mathbf{Bl}_d^k(n')$ whenever $n' \geqslant n$.

Now define a binary sequence $\boldsymbol{\lambda}^{(k)}$ such that, for each $n \in \mathbf{N}^+$:

(i) if $\lambda_n^{(k)} = 0$, then

- ❖ a prior term of $\boldsymbol{\lambda}^{(k)}$ is 1, *or*
- ❖ either n is odd and $\neg\mathbf{Bl}_0^k(n)$, or n is even and $\neg\mathbf{Bl}_1^k(n)$;

(ii) if $\lambda_n^{(k)} = 1$, then

- ❖ every prior term of $\boldsymbol{\lambda}^{(k)}$ is 0, *and*
- ❖ either n is odd and $\mathbf{Bl}_0^k(n)$, or n is even and $\mathbf{Bl}_1^k(n)$.

The sequence $\boldsymbol{\lambda}^{(k)}$ clearly has at most one term equal to 1, so invoking **LLPO**, we see that either:

- ❖ $\lambda_{2n-1}^{(k)} = 0$ for all n, in which case we set $a_{k+1} = 0$, or
- ❖ $\lambda_{2n}^{(k)} = 0$ for all n, in which case we set $a_{k+1} = 1$.

This completes the construction of α. Since B is a bar, we can find $N \in \mathbf{N}$ such that $\overline{\alpha}N \in B$. To complete the proof, we show by a backwards induction argument that B is a uniform bar. If $N = 0$, then we are done, so suppose $N \geqslant 1$. Then for each positive integer $M \leqslant N$, define the statement $S(M)$ to mean:

$$\begin{gathered}(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}M))(\exists m \leqslant N)\big[\overline{u}m \in B\big] \\ \equiv \mathbf{Bl}_{a_M}^{M-1}(N).\end{gathered}$$

Suppose that $S(M)$ holds for some positive integer $M \leqslant N$, and consider the case when $a_M = 1$ (the case $a_M = 0$ is similar). Suppose that

$$(\forall k \leqslant N)\big[\lambda_k^{(M-1)} = 0\big]. \tag{5.1}$$

If N is even, then by (i) and (5.1), we have $\neg\mathbf{Bl}_1^{M-1}(N)$ — that is,

$$(\exists u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1) * 1))(\forall m \leqslant N)\big[\overline{u}m \notin B\big].$$

But $a_M = 1$, so $\overline{\alpha}(M-1) * 1 = \overline{\alpha}M$ and this u gives a path that contradicts $S(M)$; hence N must be odd. Now since $a_M = 1$, we have $\lambda_k^{(M-1)} = 0$ for all even k, and in particular, $\lambda_{N+1}^{(M-1)} = 0$.

Then by (i) and (5.1), we have $\neg\mathbf{Bl}_1^{M-1}(N+1)$:

$$(\exists u\colon |u| = N+1 \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1) * 1))(\forall m \leqslant N+1)\left[\overline{u}m \notin B\right].$$

But now $\overline{u}N$ gives a path that contradicts $S(M)$. We therefore deduce that our supposition (5.1) was incorrect, whence there exists $j \leqslant N$ such that $\lambda_j^{(M-1)} = 1$. But we have already observed that $\lambda_k^{(M-1)} = 0$ for all even k, so j must be odd. Now by (ii), we have $\mathbf{Bl}_0^{M-1}(j)$, so certainly $\mathbf{Bl}_0^{M-1}(N)$:

$$(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1) * 0))(\exists m \leqslant N)\left[\overline{u}m \in B\right]. \tag{5.2}$$

Furthermore, since $a_M = 1$, we obtain from $S(M)$ that

$$(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1) * 1))(\exists m \leqslant N)\left[\overline{u}m \in B\right]. \tag{5.3}$$

Combining (5.2) and (5.3) yields

$$\begin{aligned}(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1)))(\exists m \leqslant N)\left[\overline{u}m \in B\right]&\\ \equiv S(M-1).&\end{aligned}$$

A similar argument leads to $S(M-1)$ in the case $a_M = 0$. Thus we have shown that $S(M)$ entails $S(M-1)$ for each positive integer $M \leqslant N$. But certainly, $S(N)$ is true; hence by N applications of *modus ponens*, we see that $S(0)$ holds — that is:

$$\begin{aligned}(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}0))(\exists m \leqslant N)\left[\overline{u}m \in B\right],&\\ \text{and so } (\forall u\colon |u| = N)(\exists m \leqslant N)\left[\overline{u}m \in B\right].&\end{aligned}$$

We therefore conclude that B is a uniform bar. □

These two proofs illustrate, by virtue of their directness, how we may actually *use* **LLPO** to recover objects of interest in the fan-theoretic setting of $\mathbf{FT}_\Delta$. In particular, given a detachable bar $B \subseteq \mathbf{2}^*$, they present two different techniques for characterising and constructing (by taking a restriction of α) a maximal-length finite path not blocked by B, and then using the existence of this path to prove that B is uniform.

We now consider the relationship between **WLPO** and $\mathbf{FT}_{\Pi^0_1}$. The following result is straightforward to show indirectly.

Proposition 5.3: $\mathbf{BISH} + \mathbf{WLPO} \ \vdash \ \mathbf{FT}_{\Pi^0_1}$.

Proof. Assume **WLPO** and let B be a Π^0_1-bar, so that

$$B = \left\{ u \in \mathbf{2}^* : \ (\forall n) \left[(u, n) \in D \right] \right\}$$

for some detachable $D \subseteq \mathbf{2}^* \times \mathbf{N}$. Fix some $u \in \mathbf{2}^*$, and define the binary sequence $(\lambda_n)_{n \geqslant 1}$ as follows:

$$\lambda_n = \begin{cases} 0 & \text{if } (u, n) \in D; \\ 1 & \text{otherwise.} \end{cases}$$

Now by **WLPO**:

$$\begin{gathered} (\forall n) \left[\lambda_n = 0 \right] \vee \neg (\forall n) \left[\lambda_n = 0 \right], \\ (\forall n) \left[(u, n) \in D \right] \vee \neg (\forall n) \left[(u, n) \in D \right], \\ u \in B \vee u \notin B. \end{gathered}$$

Hence B is a detachable bar. But

$$\mathbf{BISH} + \mathbf{LLPO} \ \vdash \ \mathbf{FT}_{\Delta}$$

and **WLPO** entails **LLPO**, so we can invoke $\mathbf{FT}_{\Delta}$ to show that B is a uniform bar. Hence $\mathbf{FT}_{\Pi^0_1}$ holds when **WLPO** is assumed. $\square$

To better understand this connection, we would ideally obtain a fully direct proof of **Proposition 5.3**; however, such a proof turns out to be hard to come by. As a step towards this goal, we present the following argument, which works in a similar manner to the second proof of **Proposition 5.1**. When combined with the foregoing observation that Π^0_1-sets are detachable under **WLPO**, this establishes $\mathbf{FT}_{\Pi^0_1}$ in a partially direct fashion.

Proposition 5.4: $\mathbf{BISH} + \mathbf{WLPO} \ \vdash \ \mathbf{FT}_{\Delta}$.

Proof. Let B be a detachable bar. If $\lambda \in B$ then we are done, so assume that $\lambda \notin B$. We use **WLPO** to construct, inductively, an infinite path $\alpha \equiv (a_n)_{n \geqslant 1}$ that begins with a maximal-length finite path not blocked by B. Suppose we have found the first k terms of α. We reprise the predicate $\mathbf{Bl}_d^k$ (which depends on $\overline{\alpha}k$) from the second proof of **Proposition 5.1** to define a binary sequence $\boldsymbol{\lambda}^{(k)}$ such that:

$$\lambda_n^{(k)} = 0 \implies (\exists i \leqslant n)\big[\mathbf{Bl}_1^k(i) \wedge \neg \mathbf{Bl}_0^k(i-1)\big] \vee (\forall i \leqslant n)\big[\neg \mathbf{Bl}_0^k(i) \wedge \neg \mathbf{Bl}_1^k(i)\big],$$
$$\lambda_n^{(k)} = 1 \implies (\exists i \leqslant n)\big[\mathbf{Bl}_0^k(i) \wedge \neg \mathbf{Bl}_1^k(i-1)\big].$$

To justify this definition, observe that, for each $n \in \mathbf{N}^+$, either

- ❖ $\neg \mathbf{Bl}_0^k(i) \wedge \neg \mathbf{Bl}_1^k(i)$ for all $i \leqslant n$, or else
- ❖ $\mathbf{Bl}_d^k(j)$ for some $d \in \mathbf{2}$ and $j \leqslant n$. Since $\lambda \notin B$, it then follows that we can find a positive integer $i \leqslant j \leqslant n$ for which $\mathbf{Bl}_d^k(i)$ and $\neg \mathbf{Bl}_d^k(i-1)$ both hold.[2] Now:
  - ◆ if $\mathbf{Bl}_{1-d}^k(i-1)$, we have $\mathbf{Bl}_{1-d}^k(i)$ and therefore $\mathbf{Bl}_{1-d}^k(i) \wedge \neg \mathbf{Bl}_d^k(i-1)$;
  - ◆ if $\neg \mathbf{Bl}_{1-d}^k(i-1)$, we have $\mathbf{Bl}_d^k(i) \wedge \neg \mathbf{Bl}_{1-d}^k(i-1)$.

Invoking **WLPO**, we have either

- ❖ $\boldsymbol{\lambda}^{(k)} = \mathbf{0}$, in which case we set $a_{k+1} = 0$; or
- ❖ $\neg\left(\boldsymbol{\lambda}^{(k)} = \mathbf{0}\right)$, in which case we set $a_{k+1} = 1$.

This completes the inductive construction of α. As in our earlier proofs, this definition captures the following informal idea: consider $\mathbf{csf}(\overline{\alpha}k)$. If the left half of this subfan is uniformly blocked by B before the right half, we grow α into the right half; if the right half is uniformly blocked before the left, we grow α into the left half.

[2] We here rely on the fact that $\mathbf{Bl}_d^k(i)$ still makes sense for $i \leqslant k$.

Compute $N \in \mathbf{N}$ with $\overline{\alpha}N \in B$. As in the second proof of **Proposition 5.1**, assume $N \geqslant 1$ and define the statement $S(M)$ to mean:

$$\begin{aligned}(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}M))(\exists m \leqslant N)\big[\overline{u}m \in B\big]\\ \equiv \mathbf{Bl}_{a_M}^{M-1}(N).\end{aligned}$$

Suppose that $S(M)$ holds for some M with $1 \leqslant M \leqslant N$. Consider a path $w = w_1 w_2 \cdots w_N$ starting with $\overline{\alpha}(M-1)$ such that $w_M \neq a_M$. Assume, with a mind to obtaining a contradiction, that

$$(\forall m \leqslant N)\big[\overline{w}m \notin B\big]. \tag{5.4}$$

We have two cases:

- **Case 1:** $a_M = 0$, so $\overline{w}M = \overline{\alpha}(M-1) * 1$. It follows from (5.4) that w is a witness to

$$\begin{aligned}(\exists u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1) * 1))(\forall m \leqslant N)\big[\overline{u}m \notin B\big]\\ \equiv \neg\mathbf{Bl}_1^{M-1}(N).\end{aligned} \tag{5.5}$$

  Furthermore, since $a_M = 0$, we have by definition, $\lambda_N^{(M-1)} = 0$; that is,

$$\begin{aligned}(\exists i \leqslant N)\big[\mathbf{Bl}_1^{M-1}(i) \wedge \neg\mathbf{Bl}_0^{M-1}(i-1)\big]\\ \vee (\forall i \leqslant N)\big[\neg\mathbf{Bl}_0^{M-1}(i) \wedge \neg\mathbf{Bl}_1^{M-1}(i)\big].\end{aligned} \tag{5.6}$$

  $S(M)$ states that $\mathbf{Bl}_0^{M-1}(N)$; hence

$$\neg(\forall i \leqslant N)\big[\neg\mathbf{Bl}_0^{M-1}(i) \wedge \neg\mathbf{Bl}_1^{M-1}(i)\big],$$

  so by (5.6), we have:

$$(\exists i \leqslant N)\big[\mathbf{Bl}_1^{M-1}(i) \wedge \neg\mathbf{Bl}_0^{M-1}(i-1)\big].$$

  So $\mathbf{Bl}_1^{M-1}(i)$ holds for some $i \leqslant N$, and we therefore have $\mathbf{Bl}_1^{M-1}(N)$. But this contradicts (5.5); hence we conclude $\neg$(5.4).

❖ **Case 2:** $a_M = 1$, so $\overline{w}M = \overline{\alpha}(M-1) * 0$. Fix any $i \in \mathbf{N}$. Two subcases arise:

- If $i > N$, then since $S(M) \equiv \mathbf{Bl}_1^{M-1}(N)$ holds, we have $\mathbf{Bl}_1^{M-1}(i-1)$.
- If $i \leqslant N$, then setting $u = \overline{w}i$, we have $\mathbf{StartsWith}(u, \overline{\alpha}(M-1) * 0)$,[3] and by (5.4), $\overline{u}m = \overline{w}m \notin B$ for all $m \leqslant i$. Hence:

$$(\exists u\colon |u| = i \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1) * 0))(\forall m \leqslant i)\left[\overline{u}m \notin B\right]$$
$$\equiv \neg\mathbf{Bl}_0^{M-1}(i).$$

We therefore have

$$(\forall i)\neg\left[\mathbf{Bl}_0^{M-1}(i) \wedge \neg\mathbf{Bl}_1^{M-1}(i-1)\right];$$

hence, $\lambda_n^{(M-1)} = 0$ for all n. But this contradicts $a_M = 1$, so again we obtain $\neg$(5.4).

In either case, we have

$$\neg(\forall m \leqslant N)\left[\overline{w}m \notin B\right],$$
$$\text{and so } (\exists m \leqslant N)\left[\overline{w}m \in B\right].$$

Since this is true for any w of length N with $\overline{w}(M-1) = \overline{\alpha}(M-1)$ and $w_M \neq a_M$, and the assumption $S(M)$ extends this to cover the case where $w_M = a_M$, we see that

$$(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}(M-1)))(\exists m \leqslant N)\left[\overline{u}m \in B\right]$$
$$\equiv S(M-1).$$

We have now shown that $S(M)$ entails $S(M-1)$ for each M, and since $S(N)$ is true, it follows that $S(0)$ holds, as in the second proof of **Proposition 5.1**:

$$(\forall u\colon |u| = N \wedge \mathbf{StartsWith}(u, \overline{\alpha}0))(\exists m \leqslant N)\left[\overline{u}m \in B\right];$$
$$\text{that is, } (\forall u\colon |u| = N)(\exists m \leqslant N)\left[\overline{u}m \in B\right].$$

So B is a uniform bar. □

[3] Again observe that this needs to (and does!) work irrespective of whether $|u| \geqslant M$ or $|u| < M$.

As in the proofs of **Proposition 5.1**, we here give a procedure for using the omniscience principle at hand to construct a maximal-length path that misses B. However, the way in which we characterise this path (and hence subsequently show that its existence forces B to be uniform) is substantially different. While we again use a backwards induction argument, as in the second proof of **Proposition 5.1**, we must employ different lines of reasoning in the cases $a_M = 0$ and $a_M = 1$.

Despite being weaker than **WLPO**, **LLPO** is in many ways easier to work with. Both of the possibilities it presents are equal in strength (and, indeed, formation). **WLPO**, on the other hand, lacks this symmetry: when applied to a binary sequence (call it λ), it presents one rather strong alternative ($\lambda = \mathbf{0}$) and one inconveniently weak alternative ($\neg(\lambda = \mathbf{0})$). The problem, then, lies in setting things up so that the desired result may be attained even in this latter case. In the case of **Proposition 5.4**, it is the detachability of the bar B that ultimately allows us to convert this negative statement into the positive conclusion that we need; hence, it is not obvious how we could modify such a proof to directly establish $\mathbf{FT}_{\Pi^0_1}$.

6 | Holomorphy (and Beyond)

We now explore some ways in which anti-Specker properties may be used to establish results about the structure of certain types of functions. The first such class is that of the *holomorphic* functions familiar from complex analysis; subsequently, we will examine more general functions that are nevertheless characterised as exhibiting a property that arises from holomorphy.

We will start with complex-valued functions defined on some inhabited region Ω of the complex plane $\mathbf{C}$. In order to characterise the behaviour of such functions in Ω, we will need some basic geometric notions. The most primitive regions we will consider are *open* and *closed balls* (or *discs*) of centre $\zeta \in \mathbf{C}$ and radius $r > 0$, denoted $B(\zeta, r)$ and $\overline{B}(\zeta, r)$ respectively, as in Chapter 3. We also follow [BB85, p. 153] in writing

$$\Gamma(\zeta, r) \equiv \left\{ z \in \mathbf{C} : \ |z - \zeta| = r \right\}$$

for the boundary of the ball $\overline{B}(\zeta, r)$.

We will typically take the domain Ω to be an *open* region in $\mathbf{C}$: a subset $X \subseteq \mathbf{C}$ is said to be open if, for each $x \in X$, there exists $r > 0$ such that $B(x, r) \subseteq X$. The complementary notion of a *closed* set is defined thus: X is *closed* if, for each convergent sequence (x_n) in X, the limit of (x_n) also belongs to X. Notice that, since $\mathbf{C}$ is complete, the closed sets within $\mathbf{C}$ are precisely those which are complete.

A totally bounded (and hence located [BV06, p. 41]) set $K \subset \mathbf{C}$ is said to be *well contained* in Ω, written $K \Subset \Omega$, if there exists $r > 0$ for which

$$K_r \equiv \left\{ z \in \mathbf{C} : \ \rho(z, K) \leqslant r \right\} \subset \Omega.$$

A function $f\colon \Omega \to \mathbf{C}$ is *holomorphic* if it can, in some sense, be represented by a power series expansion at each point in Ω;[1] we may then use these power series to analyse the behaviour of the function. It remains to make precise what we mean when we say that f can be "represented by" a power series: the notion we use is that of *uniform absolute convergence*.

A sequence $(f_n)_{n\geqslant 0}$ of functions from Ω to $\mathbf{C}$ is *uniformly convergent* if there exists a limit function $f\colon \Omega \to \mathbf{C}$ with the property that, for every $\epsilon > 0$, we can find an index $N \in \mathbf{N}^+$ such that $|f_n(z) - f(z)| \leqslant \epsilon$ for all $n \geqslant N$ and $z \in \Omega$:

$$(\forall \epsilon > 0)(\exists N)(\forall n \geqslant N)(\forall z \in \Omega)\Big[\big|f_n(z) - f(z)\big| \leqslant \epsilon\Big].$$

In the case of a series, we say that $\sum_{n=0}^{\infty} f_n$ is uniformly convergent if the sequence $(S_n)_{n\geqslant 0}$ of partial sums $S_n \equiv \sum_{i=0}^{n} f_i$ is.

This is half of the convergence property we require; to complete this definition, say that a series $\sum_{n=0}^{\infty} f_n$ is *uniformly absolutely convergent* on Ω if the series $\sum_{n=0}^{\infty}|f_n|$ is uniformly convergent on Ω.

We can now make precise our notion of holomorphy. A function $f\colon \Omega \to \mathbf{C}$ is *holomorphic* (on Ω) if

(i) f is uniformly continuous on each compact $K \Subset \Omega$,[2] and

(ii) for each point $\zeta \in \Omega$, there exist a closed ball $B \equiv \overline{B}(\zeta, R) \Subset \Omega$ and a sequence $(c_n)_{n\geqslant 0}$ of complex coefficients such that $f(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n$, with uniform absolute convergence, on B.[3]

Note that, if we have such a power series representation of f on B (that is, with uniform absolute convergence), the argument of **Theorem 2.23** of [GKR07, pp. 13–14] shows that f automatically has *pointwise* continuity on B. (We will use this fact in proving **Lemma 6.2** and **Lemma 6.13**.)

[1] Some texts refer to such functions as *analytic*, and use the term "holomorphic" to indicate complex differentiability. It is a major result of complex analysis (see, for instance, [BB85, p. 150]) that these two definitions are equivalent on appropriately related regions.

[2] As in [BB85, p. 131]. While Bishop and Bridges assume only "continuity" in their definition of complex differentiable functions, the type of continuity to which they refer is in fact uniform continuity on compact sets.

[3] Following [Car95, p. 36] and [Ahl79, pp. 35–38].

However, we will frequently need the stronger requirement (i): in particular, this guarantees the existence of infima and suprema of $|f|$ over totally bounded regions [BV06, p. 40].

The power-series definition of holomorphy we explore runs parallel to the notion of *differentiability* used by Bishop and Bridges in [BB85]. In particular, **Theorem 4.15** [BB85, p. 150] of this monograph allows one to pass from differentiability to holomorphy; accordingly, our results in this chapter are immediately applicable in that former setting. More work is required to show conversely that every holomorphic function is differentiable: although one can differentiate a power series expansion term-by-term to find a derivative on each applicable closed ball, Bishop and Bridges characterise a function f as being differentiable on a certain domain Ω only when f has the *same* derivative throughout the entirety of Ω. However, in view of their **Lemma 4.9** [BB85, p. 148], it suffices to show that, given *any* closed ball well contained in Ω, we can find a power series representation of the holomorphic function f throughout that ball (rather than simply being able to find a power series representation on *some* ball around any given point). It seems as though a result of this kind should be attainable.[4]

6.1 Isolation of Zeroes

We begin by demonstrating how the consideration of power series expansions allows us to prove two results about the location of zeroes of holomorphic functions (though perhaps "non-location of zeroes" would be nearer the mark) [BDMJ13a].

A *nonconstant* function $f\colon \Omega \to \mathbf{C}$ is one for which there exist $z, z' \in \Omega$ with $f(z) \neq f(z')$. Classically, if f is a nonconstant holomorphic function on the open region $\Omega \subseteq \mathbf{C}$, then the zeroes of f are *isolated* in the following sense: for each point ζ with $f(\zeta) = 0$, there exists $r > 0$ such that $f(z) \neq 0$ for all points z in the punctured disc $B(\zeta, r) \setminus \{\zeta\} \subset \Omega$ [SZ03, p. 339]. And indeed, continuity ensures that the same is also true in the (admittedly somewhat less interesting) case where $f(\zeta) \neq 0$.

[4] Indeed, one is to appear in [BDMJ13a].

Constructively, we may recover the following version of this principle, where, as in [BB85, p. 153], we write

$$m(f, K) \equiv \inf\{|f(z)|\colon\ z \in K\}$$

for each function $f\colon \Omega \to \mathbf{C}$ that is uniformly continuous on the totally bounded set $K \subset \Omega$.[5]

Proposition 6.1: BISH $\vdash$ Let $\Omega \subseteq \mathbf{C}$ be an inhabited open region, and fix any $\zeta \in \Omega$. Suppose that $f\colon \Omega \to \mathbf{C}$ is a holomorphic function that has a power series expansion $f(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n$, with uniform absolute convergence, on some ball $B \equiv \overline{B}(\zeta, R) \Subset \Omega$, and suppose further that $c_k \neq 0$ for some index $k > 0$. Then for each $\epsilon > 0$, there exists $r \in (0, \epsilon)$ with $m(f, \Gamma(\zeta, r)) > 0$.

We will eventually package up these hypotheses about the nonzero coefficient c_k into the concept of *local nonconstancy* — see Corollary 6.4.

Bishop and Bridges establish a stronger[6] location-of-zeroes result as **Theorem 5.11** of [BB85, pp. 157–159], the proof of which ultimately relies upon *Cauchy's integral formula*.[7] The appeal of **Proposition 6.1** is that, while it is somewhat weaker, it may be proved by directly considering the power series expansion of f. In particular, we base our proof upon the following lemma, applied iteratively.

Lemma 6.2: BISH $\vdash$ Under the hypotheses of **Proposition 6.1**, there exists an arbitrarily small $\delta > 0$ such that for each $r \in (0, \delta)$, either $c_{k'} \neq 0$ for some $k' < k$, or $m(f, \Gamma(\zeta, r)) > 0$.

Proof. Without loss of generality, take $\zeta = 0$. For $z \in B$, write

$$f(z) = \sum_{n=0}^{k-1} c_n z^n + z^k g(z),$$

where $g(z) \equiv \sum_{n=0}^{\infty} c_{k+n} z^n$ with uniform absolute convergence on B. Note that, since $g(0) = c_k \neq 0$ and g is continuous[8] at 0, we can pick some arbitrarily small $\delta \in (0, R)$ such that $|g(z)| \geqslant \frac{1}{2}|c_k| > 0$ whenever $|z| \leqslant \delta$.

Now let $r \in (0, \delta)$ and consider the sum $\sigma \equiv \sum_{n=0}^{k-1} |c_n| r^n$. Two cases arise:

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[5] Recall that the uniform continuity of f guarantees the existence of this infimum.
[6] Provided, that is, that holomorphy and differentiability are equivalent notions.
[7] Cauchy's integral formula is established as **Theorem 4.7** of [BB85, pp. 146–147].
[8] Recall **Theorem 2.23** of [GKR07, pp. 13–14].

- If $\sigma > 0$, then we are able to find some index $k' < k$ for which $c_{k'} \neq 0$.
- If $\sigma < \frac{1}{4} r^k |c_k|$, then we have, for all points $z \in \Gamma(0, r)$:

$$\begin{aligned} \left|f(z)\right| \geqslant \left|z^k g(z)\right| - \left|\sum_{n=0}^{k-1} c_n z^n\right| &\geqslant |z|^k \left|g(z)\right| - \sum_{n=0}^{k-1} |c_n| |z|^n \\ &= r^k \left|g(z)\right| - \sum_{n=0}^{k-1} |c_n| r^n \\ &> r^k \cdot \tfrac{1}{2} |c_k| - \tfrac{1}{4} r^k |c_k| = \tfrac{1}{4} r^k |c_k|. \end{aligned}$$

Hence $m(f, \Gamma(0, r)) \geqslant \frac{1}{4} |c_k| r^k > 0$. □

Proof of Proposition 6.1. Fix any $\epsilon \in (0, R)$, and write $k_0 \equiv k$ and $\delta_0 \equiv \epsilon$.

Now, given $\delta_i > 0$ and k_i with $c_{k_i} \neq 0$, we can invoke **Lemma 6.2** — replacing k by k_i — to find a number $\delta_{i+1} \in (0, \delta_i)$ with the relevant property. Fix any $r \in (0, \delta_{i+1}) \subset (0, \epsilon)$: then either $m(f, \Gamma(\zeta, r)) > 0$, in which case we are done, or else $c_{k_{i+1}} \neq 0$ for some $k_{i+1} < k_i$. In the latter case, we repeat this process: since the k_i are strictly decreasing from k, we need only iterate a maximum of k times before we either find $r \in (0, \epsilon)$ with $m(f, \Gamma(\zeta, r)) > 0$, or else see that $c_0 \neq 0$.

But if $c_0 \neq 0$, the continuity of f at ζ allows us to construct distances r, s with $0 < r < s < \epsilon$ such that $|f(z) - f(\zeta)| < \frac{1}{2}|c_0|$ for all z with $|z - \zeta| < s$. Then whenever $z \in \Gamma(\zeta, r)$, we have

$$\left|f(z)\right| \geqslant \left|c_0\right| - \left|c_0 - f(z)\right| = \left|c_0\right| - \left|f(\zeta) - f(z)\right| > \left|c_0\right| - \tfrac{1}{2}\left|c_0\right| = \tfrac{1}{2}\left|c_0\right|,$$

and therefore $m(f, \Gamma(\zeta, r)) \geqslant \frac{1}{2}|c_0| > 0$. So, in any case, the desired r exists. □

Notice that the property asserted by **Proposition 6.1** is a stronger variation upon the following notion of *local nonconstancy*: let f be a function on some inhabited open region $\Omega \subseteq \mathbf{C}$, and fix any $\zeta \in \Omega$. Then f is *locally nonconstant at* ζ if, for each $\epsilon > 0$, there exists a point $z \in \Omega$ with $|z - \zeta| < \epsilon$ and $f(z) \neq f(\zeta)$. (And if this property holds for *every* $\zeta \in \Omega$, we simply say that f is *locally nonconstant* (on Ω).) This leads us to the following observation.

Corollary 6.3: BISH $\vdash$ Let f be a function holomorphic on some inhabited open region $\Omega \subseteq \mathbf{C}$, and fix any $\zeta \in \Omega$. Then the following are equivalent.

(i) If $f(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n$ is a power series expansion for f with uniform absolute convergence on some ball $B \equiv \overline{B}(\zeta, R) \Subset \Omega$, then $c_k \neq 0$ for some index $k > 0$.

(ii) f is locally nonconstant at ζ.

Proof.

❖ (i) $\Longrightarrow$ (ii). Use the holomorphy of f to fix a power series as in (i), so that $c_k \neq 0$ for some $k > 0$. Consider the holomorphic function $g(z) = f(z) - f(\zeta)$, which has power series expansion

$$\sum_{n=0}^{\infty} c_n(z-\zeta)^n - c_0 = \sum_{n=1}^{\infty} c_n(z-\zeta)^n,$$

with uniform absolute convergence, on B. Fix any $\epsilon > 0$: applying **Proposition 6.1** to g now yields $r \in (0, \epsilon)$ with $\mu \equiv m(g, \Gamma(\zeta, r)) > 0$. Any $z \in \Gamma(\zeta, r)$ now satisfies $|z - \zeta| < \epsilon$ and $f(z) - f(\zeta) \geqslant \mu > 0$.

❖ (ii) $\Longrightarrow$ (i). Suppose that f is locally nonconstant at ζ, and let $f(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n$ be a power series expansion for f with uniform absolute convergence on some ball $B \equiv \overline{B}(\zeta, R) \Subset \Omega$. There exists a point $z \in B$ with $f(z) \neq f(\zeta)$; then,

$$0 < \left|f(z) - f(\zeta)\right| = \left|\sum_{n=1}^{\infty} c_n(z-\zeta)^n\right| \leqslant \sum_{n=1}^{\infty} \left|c_n\right|\left|z-\zeta\right|^n.$$

Write $\alpha \equiv \sum_{n=1}^{\infty} |c_n||z-\zeta|^n$. The convergence of the power series for f now allows us to find $K \in \mathbf{N}^+$ such that $\sum_{n=1}^{K} |c_n||z-\zeta|^n > \frac{1}{2}\alpha > 0$, whence $c_k \neq 0$ for some k with $1 \leqslant k \leqslant K$. □

With this in mind, we have the following alternative formulation of **Proposition 6.1**.

Corollary 6.4: BISH $\vdash$ Let f be a function holomorphic on some inhabited open region $\Omega \subseteq \mathbf{C}$, and suppose that f is locally nonconstant at $\zeta \in \Omega$. Then for each $\epsilon > 0$, there exists $r \in (0, \epsilon)$ with $m(f, \Gamma(\zeta, r)) > 0$.

Proof. Use the holomorphy of f at ζ to find a power series expansion $f(z) = \sum_{n=0}^{\infty} c_n(z-\zeta)^n$, with uniform absolute convergence, on some closed ball $B \equiv \overline{B}(\zeta, R) \Subset \Omega$. **Corollary 6.3** now allows us to find an index $k > 0$ for which $c_k \neq 0$; hence, the hypotheses of **Proposition 6.1** are met on B, and we can apply that proposition to obtain the desired result. $\square$

We now come to our second major result, which rules out a cluster point of zeroes of f at ζ in the following positive sense.

Proposition 6.5: BISH $\vdash$ Under the hypotheses of **Proposition 6.1**, let $(z_n)_{n\geqslant 1}$ be any sequence with $z_n \to \zeta$ but $z_n \neq \zeta$ for each n. Then there exists $m \in \mathbf{N}^+$ such that $f(z_m) \neq 0$.

Proof. Writing $k_0 \equiv k$ and $\delta_0 \equiv R$, we proceed in a similar manner as in the proof of **Proposition 6.1**.

Given $\delta_i > 0$ and k_i with $c_{k_i} \neq 0$, invoke **Lemma 6.2** — replacing k by k_i — to find $\delta_{i+1} \in (0, \delta_i)$ such that for each $r \in (0, \delta_{i+1})$, either $c_{k'} \neq 0$ for some $k' < k_i$ or $m(f, \Gamma(\zeta, r)) > 0$. Since $z_n \to \zeta$, we can choose an index m for which $0 < |z_m - \zeta| < \delta_{i+1}$. Then either $\mu \equiv m(f, \Gamma(\zeta, |z_m - \zeta|)) > 0$, in which case $f(z_m) \geqslant \mu > 0$ and we are done, or else $c_{k_{i+1}} \neq 0$ for some $k_{i+1} < k_i$.

Repeating this process at most k times, we either find some m for which $f(z_m) > 0$, or else see that $f(\zeta) = c_0 \neq 0$. But in this latter case, the continuity of f at ζ allows us to find $\delta \in (0, R)$ such that

$$\left|f(z)\right| \geqslant \left|c_0\right| - \left|f(z) - f(\zeta)\right| \geqslant \tfrac{1}{2}\left|c_0\right| > 0$$

whenever $|z - \zeta| < \delta$; then any m with $|z_m - \zeta| < \delta$ will give the desired result. $\square$

This result, too, provides us with a reasonably direct route for establishing the local nonconstancy of functions meeting the hypotheses of **Proposition 6.1**. Indeed, it would be straightforward to modify the proof of **Corollary 6.3** to proceed via **Proposition 6.5** rather than **Proposition 6.1**.

Corollary 6.6: BISH $\vdash$ Let f be a function holomorphic on some inhabited open region $\Omega \subseteq \mathbf{C}$, and suppose that f is locally nonconstant at $\zeta \in \Omega$. Let $(z_n)_{n\geqslant 1}$ be any sequence with $z_n \to \zeta$ but $z_n \neq \zeta$ for each n. Then there exists $m \in \mathbf{N}^+$ such that $f(z_m) \neq 0$.

Proof. An argument similar to that of **Corollary 6.4** allows us to apply **Proposition 6.5** to obtain the desired result. □

Corollary 6.7: BISH $\vdash$ Let f be a function holomorphic on some inhabited open region $\Omega \subseteq \mathbf{C}$, and suppose that f is locally nonconstant at $\zeta \in \Omega$. Then it is impossible for ζ to be a cluster point of zeroes of f.

6.2 Maximum-Modulus Theorems

We now turn our attention to the behaviour of *moduli* of holomorphic functions over regions in the complex plane. We will give particular consideration to the *norm* of such a function f over totally bounded regions $K \subset \mathbf{C}$, defined by

$$\|f\|_K \equiv \sup\left\{|f(z)|:\ z \in K\right\}$$

when f is uniformly continuous on K. (Notice that this condition is met whenever K is a subset of some compact region well contained within the (open) domain of f.)

6.2.1 Boundary and Border

In classical analysis, holomorphic functions on a region Ω attain their maximum modulus on the boundary $\partial\Omega \equiv \overline{\Omega} \cap \overline{{\sim}\Omega}$ of that region (where ${\sim}\Omega \equiv \{z \in \mathbf{C}:\ z \notin \Omega\}$) [Ahl79, p. 134].

The comprehensive development of differentiable functions given by Bishop and Bridges in [BB85] proceeds by establishing a path-integral-based concept of *analyticity*, and then using analyticity to obtain *Cauchy's integral formula*. The integral formula is subsequently used to prove the following constructive version of the foregoing maximum-modulus principle.

Proposition 6.8: BISH $\vdash$ Let f be a function differentiable on $\Omega \subseteq \mathbf{C}$, and let $K \Subset \Omega$ be a compact region with border B. Then $\|f\|_K = \|f\|_B$.

Here, a *border* for a compact set $K \subset \mathbf{C}$ is a totally bounded subset B of K such that $\overline{B}(z, \rho(z, B)) \subseteq K$ for each $z \in K$. As the next result shows, the boundary ∂K is — under appropriate restrictions on

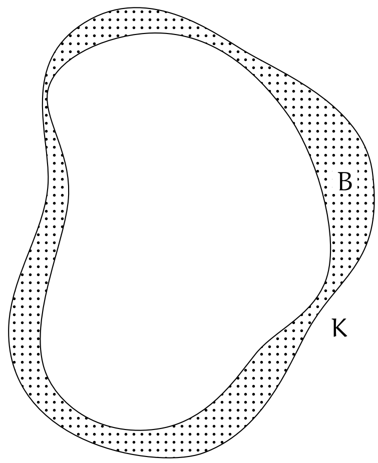

Figure 6.1: A border B obtained by swelling the boundary of K

the region K — a border for K; however, other borders are possible. The most natural of these are "swellings" of ∂K — see, for example, that depicted in **Figure 6.1**.

Proposition 6.9: BISH $\vdash$ Let $K \subset \mathbf{C}$ be a closed, located region with totally bounded boundary ∂K. Then ∂K is a border for K.

To prove this proposition, we will need some additional terminology. Let X be a Banach space. Then for endpoints $x, y \in X$, we denote by $[x, y]$ the closed segment

$$\left\{tx + (1-t)y : \ 0 \leqslant t \leqslant 1\right\} \subset X.$$

Furthermore, if $\Omega \subseteq X$ is a located set, the *metric complement* $-\Omega$ of Ω is defined thus:

$$-\Omega \equiv \left\{x \in X : \ \rho(x, \Omega) > 0\right\}.$$

We can now make use of the following result, which is **Lemma 9** of [Wan96, p. 32].

Lemma 6.10: BISH $\vdash$ Let Ω be a located subset of a Banach space X, $x_0 \in \Omega$, $y_0 \in -\Omega$ and $\epsilon > 0$. Then there exists $z \in \partial\Omega$ such that $\rho(z, [x_0, y_0]) \leqslant \epsilon$.

Proof of Proposition 6.9. Fix any point $z \in K$, and write $r \equiv \rho(z, \partial K)$. We show that $\overline{B}(z, r) \subseteq K$. Given any $\zeta_0 \in \overline{B}(z, r)$, suppose that $\rho(\zeta_0, K) > 0$ (that is, $\zeta_0 \in -K$). Writing $\gamma(t) = tz + (1-t)\zeta_0$ for $t \in [0, 1]$, take

$$\zeta_1 \equiv \gamma\left(\frac{\frac{1}{2}\rho(\zeta_0, K)}{|\zeta_0 - z|}\right) \in [z, \zeta_0]$$

(where $|\zeta_0 - z| > 0$ because $z \in K$). With a little algebra, one can verify that the following two properties hold:

- $|\zeta_1 - \zeta_0| = \frac{1}{2}\rho(\zeta_0, K)$; whence $\rho(\zeta_1, K) \geqslant \frac{1}{2}\rho(\zeta_0, K) > 0$ and so $\zeta_1 \in -K$;
- $|\zeta_1 - z| = \big||\zeta_0 - z| - \frac{1}{2}\rho(\zeta_0, K)\big|$, and since $0 < \rho(\zeta_0, K) \leqslant |\zeta_0 - z|$,

$$\left|\zeta_1 - z\right| = \left|\zeta_0 - z\right| - \tfrac{1}{2}\rho(\zeta_0, K) < \left|\zeta_0 - z\right| \leqslant r;$$

  that is, $\zeta_1 \in B(z, r)$.

Now look at the segment $[z, \zeta_1]$. Choosing any ϵ with $0 < \epsilon < r - |\zeta_1 - z|$, **Lemma 6.10** allows us to find a point $\xi \in \partial K$ with $\rho(\xi, [z, \zeta_1]) < \epsilon$, as depicted in **Figure 6.2**. Accordingly, we can find $y \in [z, \zeta_1]$ with $|\xi - y| < \epsilon$. Then:

$$\left|\xi - z\right| \leqslant \left|\xi - y\right| + \left|y - z\right| < \epsilon + \left|\zeta_1 - z\right| < r.$$

That is, ξ is a point of the boundary of K that also belongs to $B(z, r)$. Hence $\rho(z, \partial K) < r$, a contradiction. We conclude that $\rho(\zeta_0, K) = 0$ — and therefore $\zeta_0 \in K$, since K is closed — for each $\zeta_0 \in \overline{B}(z, r)$. That is, $\overline{B}(z, r) \subset K$, as was needed. □

Conversely, a border B for K need not contain the boundary ∂K: B may have "holes" along the boundary, as in **Figure 6.3**. Nevertheless, we may observe the following simple result.

Proposition 6.11: BISH $\vdash$ Let $\Omega \subseteq \mathbf{C}$ be an open region, $K \Subset \Omega$ be compact, and B be a border for K. Suppose that $f\colon \Omega \to \mathbf{C}$ is a continuous function with $\|f\|_K = \|f\|_{\partial K}$. Then $\|f\|_K = \|f\|_B$.

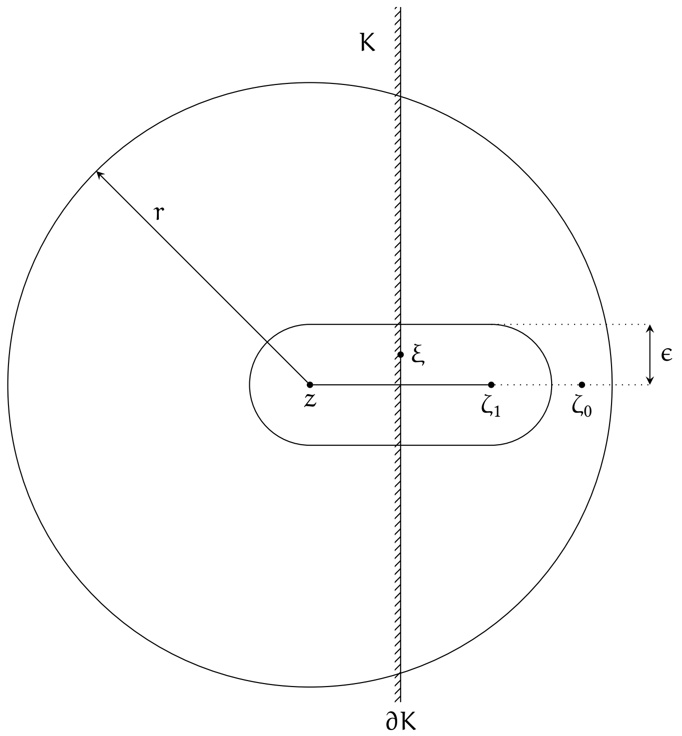

Figure 6.2: The points z, ζ_0, ζ_1 and ξ, within the ball $\overline{B}(z,r)$

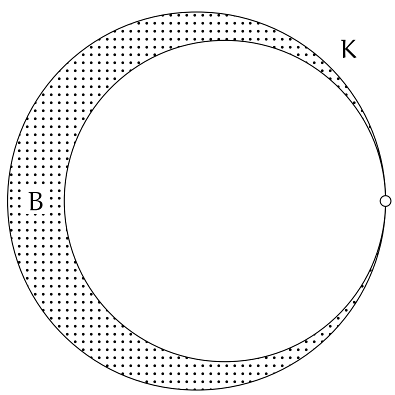

Figure 6.3: A border B that does not contain the boundary ∂K

Proof. Write $\sigma \equiv \|f\|_K$. It suffices to show that, for every $\epsilon > 0$, we can find $x \in B$ such that $|f(x)| > \sigma - \epsilon$. Fix $\epsilon > 0$. Since $\|f\|_{\partial K} = \sigma$, we can construct a point $w \in \partial K$ for which $|f(w)| > \sigma - \frac{1}{2}\epsilon$, and equivalently,

$$\Big|\sigma - \big|f(w)\big|\Big| = \sigma - \big|f(w)\big| < \tfrac{1}{2}\epsilon.$$

The continuity of f allows us to find $\delta > 0$ such that $|f(w) - f(z)| < \frac{1}{2}\epsilon$ whenever $z \in K$ is a point with $|w - z| < \delta$. Now consider the distance $r \equiv \rho(w, B)$: either $r > 0$ or $r < \delta$. If $r > 0$, consider the ball $\overline{B}(w, r)$. Since w lies on the boundary of K, this ball must contain a point not in K. But then, $\overline{B}(w, r) \nsubseteq K$, which contradicts B's being a border for K.

Hence we must have $r < \delta$; that is, we can find some point $x \in B$ for which $|w - x| < \delta$. Now, for this x,

$$\begin{aligned}\sigma - \big|f(x)\big| = \Big|\sigma - \big|f(x)\big|\Big| &\leqslant \Big|\sigma - \big|f(w)\big|\Big| + \Big|\big|f(w)\big| - \big|f(x)\big|\Big| \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.\end{aligned}$$

That is, $|f(x)| > \sigma - \epsilon$, as was required. □

6.2.2 Maximum-Modulus With Anti-Specker

As we have seen in Section 6.1, it is possible to obtain certain structural results about holomorphic functions via direct consideration of their power series expansions. The question then arises: can we recover a maximum-modulus result for holomorphic functions like that of Bishop and Bridges (that is, **Proposition 6.8**) in a similar way? At first, prospects look good: we may use a direct power series argument (bearing some resemblance to the one used to prove **Proposition 6.1**) to establish the following weak maximum-modulus property [BDMJ13a].

Proposition 6.12: BISH $\vdash$ Let f be a function holomorphic and locally nonconstant on an open region $\Omega \subseteq \mathbf{C}$. Then for each $z \in \Omega$ and $\epsilon > 0$, there exists $w \in \Omega$ such that $|w - z| < \epsilon$ and $|f(z)| < |f(w)|$.

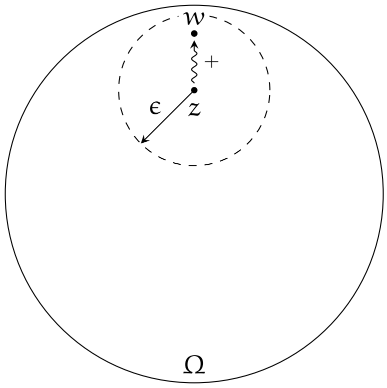

Figure 6.4: The points and distances of **Proposition 6.12**

Most of the argument for this property is encapsulated within the following lemma, which will be applied iteratively. This is based upon a nonconstructive proof of Dieudonne [Die60, pp. 207–208]; note also the similarity to **Lemma 6.2**.

Lemma 6.13: BISH $\vdash$ Under the hypotheses of **Proposition 6.1**, suppose additionally that $c_0 \neq 0$. Then either there exists a point $w \in B \equiv B(\zeta, R)$ with $|f(w)| > |f(\zeta)|$, or else $c_{k'} \neq 0$ for some k' with $1 \leqslant k' < k$.

Proof. Without loss of generality, take $\zeta = 0$. Since $k > 0$ and $c_0 \neq 0$, we can write

$$f(z) = c_0 \left(1 + \sum_{n=1}^{k-1} \frac{c_n}{c_0} z^n + \frac{c_k}{c_0} z^k + z^k h(z) \right)$$

for $z \in B$, where

$$h(z) \equiv \sum_{n=1}^{\infty} \frac{c_{k+n}}{c_0} z^n$$

with uniform absolute convergence on B. Furthermore, since $c_0, c_k \neq 0$, we can write $c_0 = r_0 e^{i\theta_0}$ and $c_k = r_k e^{i\theta_k}$ for some $r_0, r_k > 0$ and $\theta_0, \theta_k \in [0, 2\pi)$. Noting that $h(0) = 0$, continuity[9] now allows us to find $\delta \in \left(0, \frac{1}{2}R\right)$ such that $|h(z)| < \frac{r_k}{2r_0}$ whenever $|z| < 2\delta$.

Now define

$$w \equiv \delta e^{\frac{i}{k}(\theta_0 - \theta_k)}.$$

[9]Again recall **Theorem 2.23** of [GKR07, pp. 13–14].

For this choice of w, we have

$$\begin{aligned}\left|1+\frac{c_k}{c_0}w^k+w^k h(w)\right| &= \left|1+\frac{r_k}{r_0}\delta^k+w^k h(w)\right| \\ &\geqslant \left|1+\frac{r_k}{r_0}\delta^k\right| - \left|w^k h(w)\right| \\ &= 1+\frac{r_k}{r_0}\delta^k - \delta^k\left|h(w)\right| > 1+\frac{r_k}{2r_0}\delta^k.\end{aligned}$$

In order to use this inequality to recover information about f itself, we now consider the term $\sigma \equiv \sum_{n=1}^{k-1}\frac{c_n}{c_0}w^n$. Two cases arise:

- If $\sigma \neq 0$, we are able to find some index k' with $1 \leqslant k' < k$ and $c_{k'} \neq 0$.
- If $|\sigma| < \frac{r_k}{4r_0}\delta^k$, then

$$\begin{aligned}\left|f(w)\right| &\geqslant \left|c_0\right|\left(\left|1+\frac{c_k}{c_0}w^k+w^k h(w)\right| - \left|\sum_{n=1}^{k-1}\frac{c_n}{c_0}w^n\right|\right) \\ &> \left|c_0\right|\left(1+\frac{r_k}{2r_0}\delta^k-\frac{r_k}{4r_0}\delta^k\right) = \left|c_0\right|\left(1+\frac{r_k}{4r_0}\delta^k\right) > \left|c_0\right| = \left|f(0)\right|. \qquad \square\end{aligned}$$

Proposition 6.14: BISH $\vdash$ Under the hypotheses of **Proposition 6.1**, there exists a point $w \in B(\zeta, R)$ such that $|f(w)| > |f(\zeta)|$.

Proof. Applying **Proposition 6.1**, we can find $r \in (0, R)$ such that $\mu \equiv m(f, \Gamma(\zeta, r)) > 0$. Then either $|f(\zeta)| < \mu$ or $|f(\zeta)| > 0$. In the former case, choosing any $w \in \Gamma(\zeta, r)$ yields $|f(\zeta)| < \mu \leqslant |f(w)|$, completing the proof; hence we assume that $|f(\zeta)| > 0$ and therefore $c_0 \neq 0$. Write $k_0 \equiv k$.

Now, given k_i with $c_{k_i} \neq 0$, invoke **Lemma 6.13**, replacing k by k_i. This either yields a suitable point w or produces an index k_{i+1} with $1 \leqslant k_{i+1} < k_i$ and $c_{k_{i+1}} \neq 0$. Repeating this process at most k times, we eventually rule out this second possibility and hence find $w \in B(\zeta, R)$ with $|f(w)| > |f(\zeta)|$. $\square$

Proposition 6.12 quickly follows from this result.

Proof of Proposition 6.12. Fix any $z \in \Omega$ and $\epsilon > 0$. By the holomorphy of f, we can find a ball $B \equiv \overline{B}(z, R) \Subset \Omega$, with $R < \epsilon$, such that f has a power series expansion $f(x) = \sum_{n=0}^{\infty} c_n(x-z)^n$, with uniform absolute convergence, on B. In light of **Corollary 6.3**, we see that $c_k \neq 0$ for some $k > 0$; accordingly, the hypotheses of **Proposition 6.1** are met on B, and we can apply **Proposition 6.14** to find a point $w \in B$ with $|f(z)| < |f(w)|$. □

Unfortunately, if we continue working within **BISH** alone, it is not obvious how we could progress beyond this point without using the sort of approach employed by Bishop and Bridges. Indeed, this would seem to be evidence in favour of the choice they made to develop their theory of complex functions in a way largely divorced from power series expansions.

Nevertheless, if we allow ourselves to make use of the full anti-Specker property, we find that we are ultimately able to extend **Proposition 6.12** to obtain a maximum-modulus result similar to **Proposition 6.8**. The role of **AS** here is thus to *streamline* the necessary argument, rather than to bring a principle beyond **BISH** within reach. That is, it allows us to prove our desired result by making direct appeal to what are ultimately power series considerations, rather than by taking the more circuitous route of [BB85].

In order to apply an anti-Specker property, we need a sequence eventually bounded away from each point of some region. The following lemma shows how we may construct such a sequence.

Lemma 6.15: BISH $\vdash$ Let f be a function holomorphic and locally nonconstant on an open region $\Omega \subseteq \mathbf{C}$, and let $D \equiv \overline{B}(\zeta, R)$ be a closed ball well contained in Ω. Fix any $r \in (0, R)$, and write $B \equiv \overline{B}(\zeta, r)$. Let $(z_n)_{n\geqslant 1}$ be any sequence of points in D with $|f(z_n)| \to \|f\|_D$. Then (z_n) is eventually bounded away from each point of B.

Proof. Fix any point $z \in B$, and apply **Proposition 6.12** with $\epsilon \equiv R - r$. This yields a point w in the ball $B(z, R-r)$ (and thus in D) with $|f(z)| < |f(w)|$; hence, $|f(z)| < \|f\|_D$. Accordingly, choose $k \in \mathbf{N}^+$ such that $2^{-k} < \|f\|_D - |f(z)|$. The continuity of f at z yields $\delta > 0$ such that $|f(z) - f(z')| < 2^{-k-1}$ whenever $z' \in D$ is a point with $|z - z'| < \delta$.

Now find N such that $|f(z_n)| > \|f\|_D - 2^{-k-1}$ for all $n \geqslant N$. Then for such n,

$$\left|f(z_n)\right| > \left(\left|f(z)\right| + 2^{-k}\right) - 2^{-k-1} = \left|f(z)\right| + 2^{-k-1},$$

and therefore $|z - z_n| \geqslant \delta$. That is, (z_n) is eventually bounded away from z. □

This leads us to our first result capturing the spirit of the maximum-modulus property we are aiming for, if only in the rather restricted — though still very widely applicable — setting of a closed ball and its boundary.

Proposition 6.16: $\mathbf{BISH} + \mathbf{AS}^{\neg} \vdash$ Let f be a function holomorphic and locally nonconstant on an open region $\Omega \subseteq \mathbf{C}$, and let $D \equiv \overline{B}(\zeta, R)$ be a closed ball well contained in Ω. Then $\|f\|_D = \|f\|_{\partial D}$.

Proof. Suppose that $\|f\|_{\partial D} < \|f\|_D$, and write $\epsilon \equiv \|f\|_D - \|f\|_{\partial D} > 0$. Let $(z_n)_{n \geqslant 1}$ be a sequence in D with $|f(z_n)| > \|f\|_D - 2^{-n}$ for each n. Using the uniform continuity of f, compute a number $\delta \in (0, R)$ such that $|f(z) - f(z')| < \frac{1}{4}\epsilon$ whenever z and z' are points of D with $|z - z'| < \delta$. Now write $r \equiv R - \delta \in (0, R)$. Fixing any point $z \in D$, we have two possibilities:

- ❖ If $|z - \zeta| < R$, then $z \in \overline{B}(\zeta, s)$ for some $s < R$. **Lemma 6.15** then shows that (z_n) is eventually bounded away from z.
- ❖ If $|z - \zeta| > r$, consider the point

  $$b \equiv \zeta + R\frac{z - \zeta}{|z - \zeta|} \in \partial D,$$

  which has $|b - z| < \delta$. We then obtain $|f(b) - f(z)| < \frac{1}{4}\epsilon$ by our choice of δ. Now:

  $$\begin{aligned} \left|f(z)\right| &\leqslant \left|f(b)\right| + \left|f(b) - f(z)\right| \\ &< \|f\|_{\partial D} + \tfrac{1}{4}\epsilon < \|f\|_D - \tfrac{1}{2}\epsilon. \end{aligned}$$

  Since $|f(z_n)| \to \|f\|_D$, we can find $N \in \mathbf{N}^+$ such that $|f(z_n)| > \|f\|_D - \frac{1}{4}\epsilon$ for all $n \geqslant N$. Then for such n,

  $$\begin{aligned} \left|f(z_n) - f(z)\right| &\geqslant \left|f(z_n)\right| - \left|f(z)\right| \\ &> \left(\|f\|_D - \tfrac{1}{4}\epsilon\right) - \left(\|f\|_D - \tfrac{1}{2}\epsilon\right) = \tfrac{1}{4}\epsilon, \end{aligned}$$

  and therefore $|z_n - z| \geqslant \delta$. So we have again shown that (z_n) is eventually bounded away from z.

This shows that (z_n) is a Specker sequence in D. But **AS**$^{\neg}$ rules out this possibility; hence, we obtain a contradiction and conclude that $\|f\|_{\partial D} \geqslant \|f\|_D$. Since $\partial D \subset D$, it follows that these two numbers are equal. $\square$

Using the full anti-Specker property, rather than the non-Specker property, we can extend **Lemma 6.15** to the following stronger maximum-modulus result and its two corollaries, which give us plenty of computational information to work with.

Proposition 6.17: BISH + AS $\vdash$ Let f be a function holomorphic and locally nonconstant on an open region $\Omega \subseteq \mathbf{C}$, and let $D \equiv \overline{B}(\zeta, R)$ be well contained in Ω. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that if $z \in D$ and $|f(z)| > \|f\|_D - \delta$, then $\rho(z, \partial D) < \epsilon$.

Proof. Fix any $\epsilon > 0$, and write $B \equiv \overline{B}(\zeta, r)$, where $r \equiv R - \frac{1}{4}\epsilon > 0$ (assuming, without loss of generality, that $\epsilon < 4R$). We construct a sequence $(z_n)_{n \geqslant 1}$ in D as follows: consider $k \in \mathbf{N}^+$.

(i) Using uniform continuity, find a number $\alpha_k \in \left(0, \frac{1}{2}\epsilon\right)$ such that $|f(z) - f(z')| < 2^{-k-2}$ whenever z and z' are points of D with $|z - z'| < \alpha_k$. Let Y_k be a finite, B-detachable[10] α_k-approximation to D.

To construct Y_k, one could, for example, start with a β-approximation W_k to D, where $\beta \leqslant \frac{1}{4}\alpha_k$ is sufficiently small to allow for the cases where r is very near zero or R. Then for each $w \in W_k$, decide whether $\big||w - \zeta| - r\big| < \beta$ or $\big||w - \zeta| - r\big| > \frac{1}{2}\beta$, and discard w in the former case, as in **Figure 6.5**.

(ii) For each $y \in Y_k$, determine whether $|f(y)| > \|f\|_D - 2^{-k}$ or $|f(y)| < \|f\|_D - 2^{-k-1}$, and in the former case, append y to the sequence (z_n).

We can show that we will here have to add at least one term to (z_n): construct a point $y_0 \in D$ with $|f(y_0)| > \|f\|_D - 2^{-k-2}$. Then there exists $y \in Y_k$ with $|y_0 - y| < \alpha_k$. By our choice of α_k, we see that $|f(y_0) - f(y)| < 2^{-k-2}$, whence

$$\begin{aligned} \big|f(y)\big| &\geqslant \big|f(y_0)\big| - \big|f(y_0) - f(y)\big| \\ &> \left(\|f\|_D - 2^{-k-2}\right) - 2^{-k-2} \\ &= \|f\|_D - 2^{-k-1}. \end{aligned}$$

[10] That is, for all $y \in Y_k$, we can decide whether $y \in B$ or $y \notin B$.

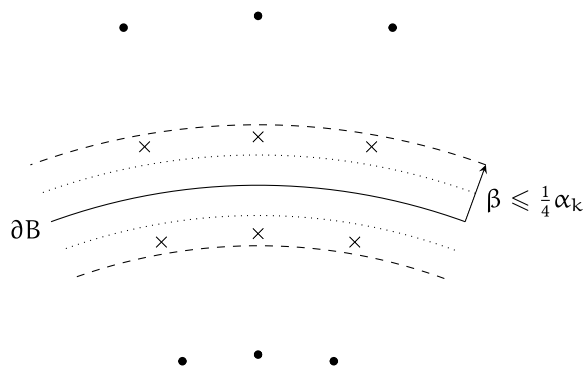

Figure 6.5: The construction of a B-detachable α_k-approximation from W_k

Given any $m \in \mathbf{N}^+$, we can repeat this process for $k = 1, 2, \ldots, m$ to find at least m terms of (z_n), after which we can determine z_m.

Observe that $|f(z_n)| \to \|f\|_D$. **Lemma 6.15** now shows that (z_n) is eventually bounded away from each point of the ball B. Since (z_n) is B-detachable, **Proposition 1** of [Bri09b, p. 439] shows that we can now apply the following equivalent version of our usual anti-Specker property:[11]

> Every B-detachable sequence in D that is eventually bounded away from each point of B is eventually not in B.

We thus obtain $N \in \mathbf{N}^+$ such that $\neg(|z_n - \zeta| < r \equiv R - \frac{1}{4}\epsilon)$ for each $n \geqslant N$. It then follows that $|z_n - \zeta| > R - \frac{1}{2}\epsilon$, and therefore $\rho(z_n, \partial D) < \frac{1}{2}\epsilon$, for such n.

Let $\delta \equiv 2^{-N-2}$, and consider any point $z \in D$ with $|f(z)| > \|f\|_D - \delta$. Find $y \in Y_N$ with $|z - y| < \alpha_N$. Then

$$\left|f(y)\right| \geqslant \left|f(z)\right| - \left|f(z) - f(y)\right| > \left\|f\right\|_D - 2^{-N-1},$$

so $y = z_\ell$ for some $\ell \geqslant N$. (We can find this ℓ by repeating the construction of (z_n).) This means we have

$$\rho(z, \partial D) < \left|z - y\right| + \rho(y, \partial D) < \alpha_N + \tfrac{1}{2}\epsilon < \epsilon. \qquad \square$$

[11] As in our proof of **Proposition 3.22**.

Corollary 6.18: BISH+AS $\vdash$ Let f be a function holomorphic and locally nonconstant on the ball $\Omega \equiv B(\zeta, R)$. Then whenever r, s are numbers with $0 < r < s < R$, we have $\|f\|_{\Gamma(\zeta,r)} < \|f\|_{\Gamma(\zeta,s)}$.

Proof. Write $D \equiv \overline{B}(\zeta, s)$ and $B \equiv \overline{B}(\zeta, r)$. Applying **Proposition 6.17**, we obtain $\delta > 0$ such that, for all points $z \in D$,

$$\left|f(z)\right| > \left\|f\right\|_D - \delta \implies \rho(z, \partial D) < s - r.$$

In particular, consider any point $z \in B \subset D$: since $\rho(z, \partial D) \geqslant s-r$, we must have $|f(z)| \leqslant \|f\|_D - \delta$. Hence $\|f\|_B \leqslant \|f\|_D - \delta < \|f\|_D$. **Proposition 6.16** now gives the desired result. $\square$

Corollary 6.19: BISH + AS $\vdash$ Let f be a function holomorphic and locally nonconstant on the ball $\Omega \equiv B(\zeta, R)$. Then f has the following property:

($\dagger$) For each $r \in (0, R)$ and $\epsilon > 0$, there exists a point $w \in \Omega$ such that $|w - \zeta| \leqslant r + \epsilon$ and $|f(w)| > \|f\|_{\overline{B}(\zeta,r)}$.

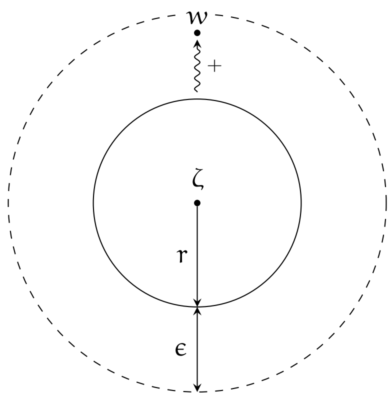

Figure 6.6: The points and distances of **Corollary 6.19**

Proof. Without loss of generality, suppose that $\epsilon < R - r$, so that the ball $D \equiv \overline{B}(\zeta, r + \epsilon)$ is well contained within Ω. Write $B \equiv \overline{B}(\zeta, r)$: by the argument of **Corollary 6.18**, we have $\|f\|_B < \|f\|_D$, and this allows us to construct a point $w \in D$ with $|f(w)| > \|f\|_B$. $\square$

Our results so far pertain to the case where the region and border under consideration are simply a closed ball and its boundary, respectively. **Proposition 6.20** now shows how our propositions of this kind (in particular, **Corollary 6.19**) may be extended to the more general case. Note that,

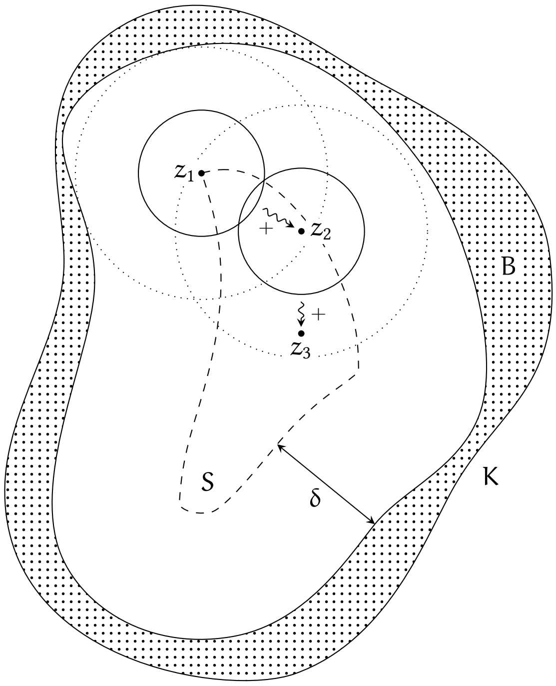

Figure 6.7: The construction used to prove **Proposition 6.20**

although the result we extend has been proved in the augmented framework **BISH** + **AS**, the extension itself is valid within **BISH** alone.

Proposition 6.20: **BISH** $\vdash$ Let f be a function holomorphic on an open region $\Omega \subseteq \mathbf{C}$, such that the property (†) holds on each open ball in Ω. Let K be a compact region well contained in Ω with border B. Then $\|f\|_K = \|f\|_B$.

We proceed by identifying a compact region $S \subseteq K$ that is separated from the border, and using the property (†) to construct a succession of overlapping discs whose centres lie within S — as depicted in **Figure 6.7** — such that the modulus of f at the centre of each disc is greater than $\|f\|$ over the central region of the preceding disc. This guarantees that these discs must continue to take up more and more space within S, rather than just overlapping indefinitely; eventually, we run out of room and thus obtain a contradiction.

Proof of Proposition 6.20. Given any $\epsilon > 0$, suppose that $\|f\|_K > \|f\|_B + \epsilon$. Using uniform continuity, compute a number $\eta > 0$ such that if z and z' are points of K with $|z - z'| < \eta$, then $|f(z) - f(z')| < \epsilon$. Let

$$P \equiv \{\rho(z, B) : \ z \in K\}.$$

The mapping $z \mapsto \rho(z, B)$ is uniformly continuous on the totally bounded set K, so by **Proposition 2.2.6** of [BV06, p. 40], its range P is also totally bounded: this allows us to compute $\sigma \equiv \sup P$. Either $\sigma > 0$ or $\sigma < \eta$. In the latter case, fix any $z \in K$. Then since $\rho(z, B) < \eta$, we can find $b \in B$ with $|z - b| < \eta$. But it then follows from our choice of η that

$$\left|f(z)\right| - \left\|f\right\|_B \leqslant \left|f(z)\right| - \left|f(b)\right| \leqslant \left|f(z) - f(b)\right| < \epsilon,$$

and therefore $|f(z)| < \|f\|_B + \epsilon$. Since $z \in K$ was arbitrary, this contradicts our assumption that $\|f\|_K > \|f\|_B + \epsilon$. Hence $\sigma > 0$.

Let g be the function $z \mapsto \sigma - \rho(z, B)$, which is uniformly continuous on K. (For, if $|z - z'| < \alpha$, then $|\rho(z, B) - \rho(z', B)| < \alpha$ and so $|g(z) - g(z')| < \alpha$.) Since $m(g, K) = 0$, **Theorem 4.9** of [BB85, p. 98] allows us to choose $\delta \in (0, \min\{\eta, \sigma\})$ such that the following region is compact:

$$S \equiv \{z \in K : \ \rho(z, B) \geqslant \delta\} \equiv \{z \in K : \ g(z) \leqslant \sigma - \delta\}.$$

Furthermore, S is inhabited: our assumption that $\|f\|_K > \|f\|_B + \epsilon$ allows us to construct a point $z_1 \in K$ with $|f(z_1)| > \|f\|_B + \epsilon$. Suppose that $\rho(z_1, B) < \delta$, and accordingly choose $b \in B$ with $|z_1 - b| < \delta$. Then by our choice of $\delta < \eta$, we have $|f(z_1) - f(b)| < \epsilon$. But

$$\left|f(z_1) - f(b)\right| \geqslant \left|f(z_1)\right| - \left|f(b)\right| > \left\|f\right\|_B + \epsilon - \left\|f\right\|_B = \epsilon,$$

so we have a contradiction. Hence $\rho(z_1, B) \geqslant \delta$; that is, $z_1 \in S$.

We base the remainder of the proof upon this region S. Let $\{y_1, \ldots, y_M\}$ be a finite $\frac{1}{4}\delta$-approximation to S, and set $N \equiv M + 1$. Observe now that if $\zeta_1, \ldots, \zeta_N$ are points of S, then there exist distinct indices $i, j \leqslant N$ such that $|\zeta_i - \zeta_j| < \frac{1}{2}\delta$. For, using the $\frac{1}{4}\delta$-approximation, there exist indices

$k_i \leqslant M$ such that $|\zeta_i - y_{k_i}| < \frac{1}{4}\delta$ for each $i \leqslant N$. By the pigeonhole principle, there exist distinct $i, j \leqslant N \equiv M + 1$ with $k_i = k_j$. Then

$$\begin{aligned} |\zeta_i - \zeta_j| &\leqslant |\zeta_i - y_{k_i}| + |y_{k_i} - \zeta_j| \\ &= |\zeta_i - y_{k_i}| + |y_{k_j} - \zeta_j| < \tfrac{1}{4}\delta + \tfrac{1}{4}\delta = \tfrac{1}{2}\delta. \end{aligned}$$

We now construct N points of S to obtain a contradiction. Assume that we have found a point $z_k \in S$ with the following properties:

(i) $|f(z_k)| > \|f\|_B + \epsilon$; and

(ii) $|f(z_k)| > \|f\|_{\overline{B}(z_n, \frac{1}{2}\delta)}$, and therefore $|z_k - z_n| \geqslant \frac{1}{2}\delta$, for all n with $1 \leqslant n < k$.

(Clearly, z_1 is such a point.) Since B is a border for K and $z_k \in S$, the disc $B(z_k, \delta)$ lies within K. We can therefore apply property (†) on this disc to find a point z_{k+1} such that $|z_{k+1} - z_k| < \delta$ and $|f(z_{k+1})| > \|f\|_{\overline{B}(z_k, \frac{1}{2}\delta)}$. In particular, $|f(z_{k+1})| > |f(z_k)| > \|f\|_B + \epsilon$, so the argument used to show that $z_1 \in S$ applies to z_{k+1}; we therefore see that this construction yields a point $z_{k+1} \in S$ satisfying properties (i) and (ii). Repeating $(N-1)$ times, we obtain points $z_1, \ldots, z_N$ of S with $|z_i - z_j| \geqslant \frac{1}{2}\delta$ for all distinct indices $i, j \leqslant N$.

But this is a contradiction in light of our choice of N. We conclude that our supposition that $\|f\|_K > \|f\|_B + \epsilon$ was incorrect; hence $\|f\|_K < \|f\|_B + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, this means that $\|f\|_K$ cannot be greater than $\|f\|_B$. Accordingly, since we know also that $\|f\|_B \leqslant \|f\|_K$, it follows that $\|f\|_B = \|f\|_K$. □

Corollary 6.21: BISH + AS $\vdash$ Let f be a function holomorphic and locally nonconstant on an open region $\Omega \subseteq \mathbf{C}$, and let $K \Subset \Omega$ be a compact region with border B. Then $\|f\|_K = \|f\|_B$.

Proof. Follows from **Corollary 6.19** and **Proposition 6.20**. □

This result comes close to establishing a maximum-modulus property analogous to the one proved by Bishop and Bridges in [BB85, p. 152]; however, their version does not require f to be locally nonconstant. It remains an open problem to somehow remove this hypothesis from **Corollary 6.21**. In order to do this, we could aim to establish the following property within **BISH** (or **BISH + AS**):

($*$) Let f be a function holomorphic and nonconstant on an open region $\Omega \subseteq \mathbf{C}$. Then f is locally nonconstant on Ω.

For then we would be able to reason like so: let $f\colon \Omega \to \mathbf{C}$ be holomorphic and suppose, as in the proof of **Proposition 6.20**, that $\|f\|_K > \|f\|_B + \epsilon$ for some $\epsilon > 0$. Then f is nonconstant, and therefore locally nonconstant by ($*$). This allows us to proceed as in the foregoing proof, ultimately obtaining a contradiction and concluding that $\|f\|_B = \|f\|_K$.

6.2.3 Maximum-Modulus With Positivity

Recall that **AS** is intuitionistically valid; hence, the maximum-modulus results of the foregoing section all hold within **INT**. We now present two alternative approaches that establish maximum-modulus theorems within this framework, using instead the weaker property **POS**. In particular, we will find it useful to apply this property to functions on arbitrary compact regions in $\mathbf{C}$: recall from **Proposition 3.7** that doing so is equivalent to using the usual positivity property on the unit interval.

First, we observe that **POS** allows us to establish a maximum-modulus theorem via a similar route to the foregoing section: that is, by proving property (†) of **Corollary 6.19**, then using **Proposition 6.20** to extend it to the more general case.

Proposition 6.22: BISH + POS $\vdash$ Let f be a function holomorphic and locally nonconstant on the ball $\Omega \equiv B(\zeta, R)$. Then f has the following property:

(†) For each $r \in (0, R)$ and $\epsilon > 0$, there exists a point $w \in \Omega$ such that $|w - \zeta| \leqslant r + \epsilon$ and $|f(w)| > \|f\|_{\overline{B}(\zeta,r)}$.

Proof. As in the proof of **Corollary 6.19**, take $\epsilon < R - r$, so that the ball $D \equiv \overline{B}(\zeta, r + \epsilon)$ is well contained within Ω, and write $B \equiv \overline{B}(\zeta, r)$. Let $g\colon B \to \mathbf{R}$ be the function $z \mapsto \|f\|_D - |f(z)|$. Since f is uniformly continuous on B, so is g. Furthermore, given any point $z \in B$, we can apply **Proposition 6.12** to obtain a point $z' \in D$ with $|f(z)| < |f(z')|$; hence, $|f(z)| < \|f\|_D$. That is, g is positive-valued on B; invoking **POS**, we obtain $\mu \equiv \inf g > 0$. But we can now construct $w \in D$ with $|f(w)| > \|f\|_D - \frac{1}{2}\mu$, and since

$$\begin{aligned}\mu &\equiv \inf\left\{\|f\|_D - |f(z)|:\ z \in B\right\}\\ &= \|f\|_D - \sup\left\{|f(z)|:\ z \in B\right\}\\ &= \|f\|_D - \|f\|_B > 0,\end{aligned}$$

it follows that $|f(w)| > \frac{1}{2}\left(\|f\|_D + \|f\|_B\right) > \|f\|_B$. □

Corollary 6.23: BISH + POS $\vdash$ Let f be a function holomorphic and locally nonconstant on an open region $\Omega \subseteq \mathbf{C}$, and let $K \Subset \Omega$ be a compact region with border B. Then $\|f\|_K = \|f\|_B$.

Proof. Follows from **Proposition 6.22** and **Proposition 6.20**. □

But in fact, positivity allows us to dispense with the requirement that f be locally nonconstant, and produce instead the following reasonably direct yet fully general maximum-modulus result.

Proposition 6.24: BISH + POS $\vdash$ Let f be a function holomorphic on an open region $\Omega \subseteq \mathbf{C}$, and let $K \Subset \Omega$ be a compact region with border B. Then $\|f\|_K = \|f\|_B$.

Proof. Suppose that $\|f\|_B < \|f\|_K$, and write

$$\epsilon \equiv \|f\|_K - \|f\|_B > 0.$$

Using the uniform continuity of f, find $\eta > 0$ such that $|f(z) - f(z')| < \frac{1}{2}\epsilon$ whenever z and z' are points of K with $|z - z'| < \eta$. Now, as in the proof of **Proposition 6.20**, compute a number $\delta \in (0, \eta)$ such that the set

$$S \equiv \left\{z \in K:\ \rho(z, B) \geqslant \delta\right\}$$

is inhabited and compact.

We proceed by a construction similar to that of **Proposition 6.22**. Let $g\colon S \to \mathbf{R}$ be the uniformly continuous function $z \mapsto \|f\|_K - |f(z)|$. Now for any point $z \in S$, we can apply **Proposition 6.12** to obtain a point w in the ball $D \equiv B(z, \delta)$ with $|f(z)| < |f(w)|$; then, since B is a border for K, we have $w \in D \subset K$ and therefore see that $|f(z)| < \|f\|_K$. Hence $g(z) > 0$ for all $z \in S$. Accordingly, we can apply **POS** on S to deduce that $\inf g > 0$ and therefore $\|f\|_K > \|f\|_S$.

Now let $(z_n)_{n \geqslant 1}$ be a sequence in K with $|f(z_n)| > \|f\|_K - 2^{-n}$ for each n. Then since $\|f\|_K > \|f\|_S$, we must have $z_n \notin S$ — and thus $\rho(z_n, B) < \eta$ — eventually. But then we can find $b_n \in B$ with $|z_n - b_n| < \eta$ and therefore $|f(z_n) - f(b_n)| < \frac{1}{2}\epsilon$. Hence

$$\begin{aligned}
\big|f(z_n)\big| &\leqslant \big|f(z_n) - f(b_n)\big| + \big|f(b_n)\big| \\
&< \tfrac{1}{2}\epsilon + \|f\|_B \\
&= \tfrac{1}{2}\epsilon + \big(\|f\|_K - \epsilon\big) = \|f\|_K - \tfrac{1}{2}\epsilon
\end{aligned}$$

for all sufficiently large n, which is a contradiction, since we have constructed (z_n) so that $|f(z_n)| \to \|f\|_K$. We therefore conclude that our assumption that $\|f\|_B < \|f\|_K$ was false, whence we have $\|f\|_B \geqslant \|f\|_K$ and thus that $\|f\|_K = \|f\|_B$, as required. □

6.3 Zero-Stability

We now present a pair of results that explore how anti-Specker properties may be used to recover information about how functions of a certain kind are (or are not) structured with respect to the location of their zeroes [BDMJ13c]. Accordingly, for functions f from a metric space X into a normed space Y, we shall be concerned with the *zero set*

$$\mathcal{Z}_f \equiv \big\{x \in X\colon\ f(x) = 0\big\}.$$

We move to a more general setting than that of the holomorphic functions, but nevertheless retain some aspect of their form. In particular notice that, provided holomorphic functions are indeed differentiable in the sense of Bishop and Bridges [BB85, pp. 130–131], we can apply a strong location-of-zeroes result of theirs to prove the following proposition.

Proposition 6.25: BISH $\vdash$ Let $\Omega \subseteq \mathbf{C}$ be an open region, $K \Subset \Omega$ be compact, and B be a border for K. Suppose that $f\colon \Omega \to \mathbf{C}$ is a holomorphic function with $m(f, B) > m(f, K)$. Then f satisfies the following property:

($*$) For every $\epsilon > 0$, we can find $\delta > 0$ such that, for each $z \in K$: if $|f(z)| < \delta$, then $|z - \zeta| < \epsilon$ for some zero $\zeta \in \mathcal{Z}_f$.

Proof. Since $m(f, B) > 0$ and $m(f, K) \neq m(f, B)$, **Corollary 5.3** of [BB85, p. 153] shows that $m(f, K) = 0$. We are now able to apply **Theorem 5.11** of the same to find points $z_1, \dots, z_n \in K$ and a function g differentiable on K such that $\mu \equiv m(g, K) > 0$ and

$$f(z) = (z - z_1) \cdots (z - z_n)\, g(z)$$

for $z \in K$. Fix any $\epsilon > 0$, and observe that

$$\left|z - z_1\right| \cdots \left|z - z_n\right| \leqslant \frac{\left|f(z)\right|}{\mu}.$$

Accordingly, if we take $\delta \equiv \mu \cdot \left(\frac{1}{2}\epsilon\right)^n$, we get

$$\left|z - z_1\right| \cdots \left|z - z_n\right| < \left(\tfrac{1}{2}\epsilon\right)^n$$

whenever $|f(z)| < \delta$. This rules out the possibility of having $|z - z_i| > \frac{1}{2}\epsilon$ for all i; hence we are able to find an index k with $|z - z_k| < \epsilon$. $\square$

We will henceforth focus our attention upon functions that exhibit a pointwise variant of the property ($*$). Let X be a metric space and Y a normed space. We say that $f \colon X \to Y$ is *zero-stable* (on X) if, for each point $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $\|f(x)\| < \delta$, then $\rho(x, \zeta) < \epsilon$ for some $\zeta \in \mathcal{Z}_f$: that is,

$$(\forall x \in X)(\forall \epsilon > 0)(\exists \delta > 0)\Big[\big\|f(x)\big\| < \delta \implies (\exists \zeta \in \mathcal{Z}_f)\big[\rho(x, \zeta) < \epsilon\big]\Big].$$

We will also consider how we may attain a stronger generalisation of ($*$), in its original uniform formulation. The function f is said to be *uniformly zero-stable* (on X) if, for each $\epsilon > 0$, we can find $\delta > 0$ such that, for each $x \in X$: if $\|f(x)\| < \delta$, then $\rho(x, \zeta) < \epsilon$ for some $\zeta \in \mathcal{Z}_f$. That is,

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in X)\Big[\big\|f(x)\big\| < \delta \implies (\exists \zeta \in \mathcal{Z}_f)\big[\rho(x, \zeta) < \epsilon\big]\Big].$$

In the classical setting, it is straightforward to show that every function $f\colon X \to Y$ is zero-stable. For, given $x \in X$ and $\epsilon > 0$, we have either $f(x) = 0$ or $f(x) \neq 0$. In the former case, set $\delta = 1$, and choose $\zeta = x$ to satisfy the consequent of

$$\left\|f(x)\right\| < \delta \implies (\exists \zeta \in \mathcal{Z}_f)\left[\rho(x, \zeta) < \epsilon\right];$$

in the latter case, choose $\delta = \|f(x)\| > 0$ to falsify the antecedent.

In fact, we can use a sequential compactness argument to obtain the following more interesting result.

Proposition 6.26: CLASS $\vdash$ Let X be a compact metric space, and let f be a sequentially continuous mapping of X into a normed space Y. Then f is uniformly zero-stable.

Proof. Fix any $\epsilon > 0$, and suppose

$$(\forall \delta > 0)(\exists x \in X)\Big[\left\|f(x)\right\| < \delta \wedge (\forall \zeta \in \mathcal{Z}_f)\left[\rho(x, \zeta) \geqslant \epsilon\right]\Big]. \tag{6.1}$$

This gives us a sequence $(x_n)_{n \geqslant 1}$ of points in X such that, for each n, $\|f(x_n)\| < 2^{-n}$, and $\rho(x_n, \zeta) \geqslant \epsilon$ for each $\zeta \in \mathcal{Z}_f$. Use the sequential compactness of X to find a subsequence $(x_{n_k})_{k \geqslant 1}$ that converges to some point $x \in X$. Since $\|f(x_{n_k})\| \to 0$, we must have $\|f(x)\| = 0$ by sequential continuity; that is, $x \in \mathcal{Z}_f$. But by our definition of (x_n), this now gives us $\rho(x_{n_k}, x) \geqslant \epsilon$ for all $k \in \mathbf{N}^+$ — a contradiction. So (6.1) must be false, and uniform zero-stability now follows. □

Constructively, the situation is somewhat less straightforward. While holomorphic functions still exhibit zero-stability on appropriate domains (in light of **Proposition 6.25**), continuous functions in our more general setting need not. Indeed, even the claim that functions in a class as restricted as the quadratic polynomials of a real variable are zero-stable entails **LPO**.

To see this, fix any $\alpha \geqslant 0$, and consider the quadratic function $f\colon [0, 1] \to \mathbf{R}$ defined by $f(x) = x^2 + \alpha$. Suppose that f is zero-stable. In particular, we can apply zero-stability at 0 to obtain a number $\delta > 0$ such that if $\alpha = |f(0)| < \delta$, then there exists $\zeta \in [0, 1]$ with $f(\zeta) = 0$ and $|\zeta| < 1$. Now, either $\alpha > 0$ or $\alpha < \delta$. In the latter case: if $\alpha > 0$, we get $f(\zeta) = \zeta^2 + \alpha > 0$ for all real ζ with $|\zeta| < 1$, a contradiction; hence, $\alpha = 0$. So zero-stability allows us to decide whether $\alpha > 0$ or $\alpha = 0$, a decision which is equivalent to **LPO** [BV06, p. 29].

With this in mind, we will *assume* (pointwise) zero-stability in the results that follow. The focus, then, is upon how we can combine zero-stability with anti-Specker properties to obtain structural information about the functions in question. We begin by showing how $\mathbf{AS}^{\mathbf{ltd}}$ allows us to pass from zero-stability to uniform zero-stability, provided $\mathcal{Z}_f$ is inhabited and separable. One can see by comparison with **Proposition 6.26** that this is another instance in which an anti-Specker property allows us to recapture some of the power of sequential compactness.

Proposition 6.27: $\mathbf{BISH} + \mathbf{AS}^{\mathbf{ltd}} \vdash$ Let X be a compact metric space, and let f be a zero-stable, continuous mapping of X into a normed space Y such that $\mathcal{Z}_f$ is inhabited and separable. Then f is uniformly zero-stable on X.

Proof. Fix any $\epsilon > 0$. Since $\mathcal{Z}_f$ is inhabited, we can use **Theorem 4.9** of [BB85, p. 98] to construct a strictly decreasing sequence of numbers $\eta_n \in (0, 2^{-n})$ for which the sets

$$S_n \equiv \left\{x \in X\colon\ \left\|f(x)\right\| \leqslant \eta_n\right\}$$

are compact. Now let $(z_n)_{n\geqslant 1}$ be a dense sequence in $\mathcal{Z}_f$, and write

$$Z_n \equiv \left\{z_1, z_2, \ldots, z_n\right\} \subseteq \mathcal{Z}_f$$

for each n. Since Z_n is finite and therefore located in X, we can form the set

$$P_n \equiv \left\{\rho(x, Z_n)\colon\ x \in S_n\right\} \subset \mathbf{R}.$$

As in the proof of **Proposition 6.20**, P_n is totally bounded: we thus see that $\sup P_n$ exists. This allows us to construct a binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\begin{aligned} \lambda_n = 0 &\implies \sup P_n > \tfrac{1}{2}\epsilon, \text{ and} \\ \lambda_n = 1 &\implies \sup P_n < \epsilon. \end{aligned}$$

Let $X \cup \{\omega\}$ be a one-point extension of X, and use (λ_n) to define a sequence $(x_n)_{n\geqslant 1}$ in this one-point extension as follows:

- if $\lambda_n = 0$, pick $x_n \in S_n$ so that $\rho(x_n, Z_n) > \frac{1}{2}\epsilon$;
- if $\lambda_n = 1$, set $x_n = \omega$.

We now show that (x_n) is eventually bounded away from each point of X. Fix any $x \in X$. The zero-stability of f gives us a number $\alpha > 0$ such that

$$\|f(x)\| < \alpha \implies (\exists \zeta \in \mathcal{Z}_f)\left[\rho(\zeta, x) < \tfrac{1}{4}\epsilon\right]. \tag{6.2}$$

Two possibilities arise. Note that if $\lambda_n = 1$, we have $\rho(x, x_n) \geqslant \rho(\omega, X) > 0$, which means that x_n is immediately bounded away from x; hence, we need only consider the situation where $\lambda_n = 0$.

- If $\|f(x)\| < \alpha$, then (6.2) gives us a zero $\zeta \in \mathcal{Z}_f$ with $\rho(\zeta, x) < \frac{1}{4}\epsilon$. Choose $N \in \mathbf{N}^+$ such that $\rho(\zeta, z_N) < \frac{1}{4}\epsilon - \rho(\zeta, x)$, and set $\delta = \frac{1}{4}\epsilon$. Then for all $n \geqslant N$ with $\lambda_n = 0$, we have $z_N \in Z_n$ and thus $\rho(x_n, z_N) > \frac{1}{2}\epsilon$; it follows that

  $$\rho(x_n, x) \geqslant \rho(x_n, z_N) - \big(\rho(z_N, \zeta) + \rho(\zeta, x)\big) > \tfrac{1}{2}\epsilon - \tfrac{1}{4}\epsilon = \tfrac{1}{4}\epsilon = \delta.$$

- If $\|f(x)\| > 0$, choose $N \in \mathbf{N}^+$ so that $\eta_N < \frac{1}{2}\|f(x)\|$, and then use continuity to choose $\delta > 0$ so that $\|f(x) - f(x')\| < \frac{1}{2}\|f(x)\|$ whenever $x' \in X$ is a point with $\rho(x, x') < \delta$. Then for all $n \geqslant N$ with $\lambda_n = 0$, we have $x_n \in S_n$, and so

  $$\begin{aligned}\big\|f(x) - f(x_n)\big\| &\geqslant \big\|f(x)\big\| - \big\|f(x_n)\big\| \\ &\geqslant \big\|f(x)\big\| - \eta_n \\ &\geqslant \big\|f(x)\big\| - \eta_N \\ &> \big\|f(x)\big\| - \tfrac{1}{2}\big\|f(x)\big\| = \tfrac{1}{2}\big\|f(x)\big\|.\end{aligned}$$

  Hence $\rho(x, x_n) \geqslant \delta$.

In either case, (x_n) is eventually bounded away from x. The limited anti-Specker property now gives us an index k for which $x_k = \omega$, whence $\lambda_k = 1$. Then, for each $x \in X$ with $\|f(x)\| < \eta_k$, we have $x \in S_k$ and therefore $\rho(x, Z_k) < \epsilon$. That is, there exists a zero $\zeta \in Z_k \subseteq \mathcal{Z}_f$ with $\rho(x, \zeta) < \epsilon$; writing $\eta \equiv \eta_k$, we have shown that

$$(\forall \epsilon > 0)(\exists \eta > 0)(\forall x \in X)\Big[\big\|f(x)\big\| < \eta \implies (\exists \zeta \in \mathcal{Z}_f)\big[\rho(x, \zeta) < \epsilon\big]\Big]. \qquad \square$$

Note that this result also applies in the case where $\mathcal{Z}_f$ is *located* in X. For, **Proposition 2.2.4** of [BV06, p. 39] shows that X is separable, and located subsets of a separable metric space are themselves separable.

For our next result, we introduce the following notion: a subset S of a metric space X is said to be *countably isolated* if there exists a one-one enumeration $(s_n)_{n\geqslant 1}$ of S that is eventually bounded away from each of its own terms. It is straightforward to show that, if X is sequentially compact and f is a sequentially continuous mapping from X into a normed space, then $\mathcal{Z}_f$ cannot be countably isolated. The following proposition uses the non-Specker property to help recapture this idea in a semi-constructive setting.

Proposition 6.28: BISH + AS$^-$ $\vdash$ Let X be a compact metric space, and let f be a zero-stable, continuous mapping of X into a normed space Y. Then $\mathcal{Z}_f$ is not countably isolated.

Proof. Suppose that $\mathcal{Z}_f$ is countably isolated, and let $(z_n)_{n\geqslant 1}$ be a one-one enumeration of $\mathcal{Z}_f$ that is eventually bounded away from each term z_k. Fix any point $x \in X$: we show that (z_n) is eventually bounded away from x. Use the zero-stability of f at x to find a sequence of positive distances $(\eta_n)_{n\geqslant 1}$ such that, for each n,

$$\big\|f(x)\big\| < \eta_n \implies (\exists \zeta \in \mathcal{Z}_f)\big[\rho(x, \zeta) < 2^{-n}\big]. \tag{6.3}$$

Now define a binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \implies \big\|f(x)\big\| < \eta_n, \text{ and}$$
$$\lambda_n = 1 \implies \big\|f(x)\big\| > 0.$$

Observe that if $\lambda_k = 1$ for some $k \in \mathbf{N}^+$, it follows that (z_n) is bounded away from x. For if $\|f(x)\| > 0$, the continuity of f allows us to compute $\delta > 0$ such that $\|f(x')\| > 0$ whenever $\rho(x, x') < \delta$. Then for each n, $\|f(z_n)\| = 0$ and so $\rho(x, z_n) \geqslant \delta$.

With this in mind, assume that $\lambda_1 = 0$ (for if $\lambda_1 = 1$, we are done). Construct a sequence $(\zeta_n)_{n \geqslant 1}$ in X as follows: for each $n \in \mathbf{N}^+$,

- ❖ if $\lambda_m = 0$ for all $m \leqslant n$, use (6.3) to choose $\zeta_n \in \mathcal{Z}_f$ such that $\rho(x, \zeta_n) < 2^{-n}$;
- ❖ if $\lambda_m = 1$ for some minimal $m \leqslant n$, set $\zeta_n = \zeta_{m-1}$.

Then (ζ_n) is a Cauchy sequence and therefore converges to some limit $\zeta \in X$. Since each term of (ζ_n) is a zero of f, we have $f(\zeta) = 0$ by (sequential) continuity. That is, ζ is equal to a term of (z_n); hence (z_n) must be eventually bounded away from ζ.

Accordingly, find $N_0 \in \mathbf{N}^+$ and $\delta > 0$ such that $\rho(z_n, \zeta) > 2\delta$ for all $n \geqslant N_0$. Consider the distance $d \equiv \rho(\zeta, x)$.

- ❖ If $d < \delta$, then for all $n \geqslant N_0$:

$$\rho(z_n, x) \geqslant \rho(z_n, \zeta) - \rho(\zeta, x) > 2\delta - \delta = \delta.$$

- ❖ If $d > 0$, then we can find an index N_1 such that $2^{-N_1} < \frac{1}{2}d$ and $\rho(\zeta_n, \zeta) \leqslant \frac{1}{2}d$ for all $n \geqslant N_1$. Then for all such n, we have

$$\rho(\zeta_n, x) \geqslant \rho(\zeta, x) - \rho(\zeta_n, \zeta) \geqslant d - \tfrac{1}{2}d = \tfrac{1}{2}d. \tag{6.4}$$

  If it were the case that $\lambda_m = 0$ for all $m \leqslant N_1$, we would have $\rho(\zeta_{N_1}, x) < 2^{-N_1} < \frac{1}{2}d$, contradicting (6.4) for $n = N_1$. Hence $\lambda_k = 1$ for some $k \leqslant N_1$.

In either case, we see that (z_n) is eventually bounded away from the point x; since x was arbitrary, (z_n) is a Specker sequence in X. This is a contradiction in view of $\mathbf{AS}^-$. □

For future work, it would be interesting to know whether this result can be adapted to rule out the countable isolation of $\mathcal{Z}_f$ in some *positive* sense, ideally using the limited anti-Specker property (but possibly falling back upon the full anti-Specker property instead).

Envoi

We thus arrive at the end of our investigation. We have identified and classified a family of several weak semi-constructive principles, the *anti-Specker properties*, and subsequently illustrated where and how these principles may be used. Notably, they may sometimes be used in the stead of the highly nonconstructive property of *sequential compactness*; however, even when not applied to compactness results, they often allow us to recover information about such things as the structure of certain classes of functions.

Prominently, we have seen how the *limited* and *increasing anti-Specker properties*, $\mathbf{AS}^{\mathbf{ltd}}$ and $\mathbf{AS}^{\uparrow}$, allow us to establish a pair of Heine-Borel compactness results — one of which links $\mathbf{AS}^{\mathbf{ltd}}$ to Brouwer's fan theorem for detachable bars, $\mathbf{FT}_{\Delta}$. Subsequently, we gave new, direct proofs to illuminate the equivalence of the *non-Specker property* $\mathbf{AS}^{\neg}$ and a family of weak, negative fan-theoretic principles. Alongside these results, we have classified the similar-in-principle *limit-stability property* **LSP** and several recursively-valid antitheses of intuitionistic principles.

In a similar capacity, we have examined the omniscience principles **WLPO** and **LLPO**, presenting direct proofs that explore their relationships with $\mathbf{FT}_{\Delta}$, $\mathbf{AS}^{\neg}$ and **LSP**.

As well as allowing one to prove results not attainable within **BISH** alone, our anti-Specker properties may also be used to streamline proofs of known results. In particular, we have seen how they may be used to recover a version of the *maximum modulus theorem* for holomorphic functions, based upon a power-series argument rather than the use of Cauchy's integral formula (as was the approach of [BB85]).

Finally, we showed how $\mathbf{AS}^{\mathbf{ltd}}$ and $\mathbf{AS}^{\neg}$ can be used to recover information about the structure of zeroes for so-called *zero-stable* functions.

In conclusion, then: these anti-Specker properties are a valuable aid in semi-constructively establishing important results of analysis. But as suggested throughout the preceding chapters, there remains plenty of further work to be done. This is typically indicated at the relevant places; however, a common theme concerns our many results of the form

$$\mathbf{BISH} + \mathbf{P} \;\vdash\; \mathbf{Q}$$

(**Corollary 3.10** is a prominent example). In many cases, it would be desirable to expand upon these results either by reversing the implication (thereby showing that **P** and **Q** are equivalent over **BISH**, or maybe over **BISH** plus some additional principle) or by giving a model of **BISH** in which **Q** holds but **P** does not (thereby showing that the implication is strict).

Christchurch, December 2013 James E. Dent

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